JORDANIAN DOUBLE EXTENSIONS OF A QUADRATIC VECTOR SPACE AND SYMMETRIC NOVIKOV ALGEBRAS

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ABSTRACT. First, we study pseudo-Euclidean Jordan algebras obtained as double extensions of a quadratic vector space by a one-dimensional algebra. We give an isomorphic characterization of 2-step nilpotent pseudo-Euclidean Jordan algebras. Next, we find a Jordan-admissible condition for a Novikov algebra $\mathfrak{N}$. Finally, we focus on the case of a symmetric Novikov algebra and study it up to dimension 7.

0. INTRODUCTION

All algebras considered in this paper are finite-dimensional algebras over $\mathbb{C}$. The general framework for our study is the following: let $\mathfrak{q}$ be a complex vector space equipped with a non-degenerate bilinear form $B_{\mathfrak{q}}$ and $C : \mathfrak{q} \to \mathfrak{q}$ be a linear map. We associate a vector space

$$\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$$

to the triple $(\mathfrak{q}, B_{\mathfrak{q}}, C)$ where $(\mathfrak{t} = \mathrm{span}\{x_1, y_1\}, B_{\mathfrak{t}})$ is a 2-dimensional vector space and $B_{\mathfrak{t}} : \mathfrak{t} \times \mathfrak{t} \to \mathbb{C}$ is the bilinear form defined by

$$B_{\mathfrak{t}}(x_1, x_1) = B_{\mathfrak{t}}(y_1, y_1) = 0,\ B_{\mathfrak{t}}(x_1, y_1) = 1.$$

Define a product $\star$ on the vector space $\mathfrak{J}$ such that $\mathfrak{t}$ is a subalgebra of $\mathfrak{J}$,

$$y_1 \star x = C(x),\ x_1 \star x = 0,\ x \star y = B_{\mathfrak{q}}(C(x), y)x_1$$

for all $x, y \in \mathfrak{q}$ and such that the bilinear form $B_{\mathfrak{J}} = B_{\mathfrak{q}} + B_{\mathfrak{t}}$ is *associative* (that means $B_{\mathfrak{J}}(x \star y, z) = B_{\mathfrak{J}}(x, y \star z),\ \forall x, y, z \in \mathfrak{J}$). We call $\mathfrak{J}$ is a *double extension of* $\mathfrak{q}$ *by* C. It can be completely characterized by the pair $(B_{\mathfrak{q}}, C)$ combined with some properties of the 2-dimensional subalgebra $\mathfrak{t}$.

A rather interesting note is that such algebras $\mathfrak{J}$ can also be classified up to isometric isomorphisms (or i-isomorphisms, for short) or isomorphisms. This is successfully done for the case of $B_{\mathfrak{q}}$ symmetric or skew-symmetric, C skew-symmetric (with respect to $B_{\mathfrak{q}}$) and $B_{\mathfrak{t}}$ symmetric (see [FS87], [DPU] and [Duo10]). In these cases, a double extension of $\mathfrak{q}$ by C is a quadratic Lie algebra or a quadratic Lie superalgebra. Their classification is connected to the well-known classification of adjoint orbits in classical Lie algebras theory[CM93]. That is, there is a one-to-one

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correspondence between isomorphic classes of those algebras and adjoint G-orbits in $\mathbb{P}^1(\mathfrak{g})$, where G is the isometry group of $B_{\mathfrak{q}}$ and $\mathbb{P}^1(\mathfrak{g})$ is the projective space associated to the Lie algebra $\mathfrak{g}$ of G. Therefore, it is natural to consider similar algebras corresponding to the remaining different cases of the pair $(B_{\mathfrak{q}}, C)$.

Remark that the above definition of a double extension is a special case of a one-dimensional extension in terms of the double extension notion initiated by V. Kac to construct quadratic solvable Lie algebras [Kac85]. This notion is generalized effectively for quadratic Lie algebras [MR85] and many other non-anticomutative algebras (see [BB99], [BB] and [AB10]) to obtain an inductive characterization (also called *generalized double extension*). Unfortunately, the classification (up to isomorphisms or i-isomorphisms) of the algebras obtained by the double extension or generalized double extension method seems very difficult, even in nilpotent or low dimensional case. For example, nilpotent pseudo-Euclidean Jordan algebras up to dimension 5 are listed completely but only classified in cases up to dimension 3 [BB].

In Section 2, we apply the work of A. Baklouti and S. Benayadi in [BB] for the case of a one-dimensional double extension of the pair $(B_{\mathfrak{q}}, C)$ to obtain pseudo-Euclidean (commutative) Jordan algebras (i.e. Jordan algebras endowed with a non-degenerate associative symmetric bilinear form). Consequently, the bilinear forms $B_{\mathfrak{q}}$, $B_{\mathfrak{t}}$ are symmetric, C must be also symmetric (with respect to $B_{\mathfrak{q}}$) and the product $\star$ is defined by:

$$\begin{aligned}&(x+\lambda x_1+\mu y_1)\star(y+\lambda' x_1+\mu' y_1) := \\ &\mu C(y)+\mu' C(x)+B_{\mathfrak{q}}(C(x),y)x_1+\varepsilon\left(\left(\lambda\mu'+\lambda'\mu\right)x_1+\mu\mu' y_1\right),\end{aligned}$$

$\varepsilon \in \{0,1\}$, for all $x, y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$.

Since there exist only two one-dimensional Jordan algebras, one Abelian and one simple, then we have two types of extensions called respectively *nilpotent double extension* and *diagonalizable double extension*. The first result (Proposition 2.1, Corollary 2.2, Corollary 2.7 and Appendix) is the following:

THEOREM 1:

(1) *If $\mathfrak{J}$ is the nilpotent double extension of $\mathfrak{q}$ by C then $C^3 = 0$, $\mathfrak{J}$ is 3-step nilpotent and $\mathfrak{t}$ is an Abelian subalgebra of $\mathfrak{J}$.*
(2) *If $\mathfrak{J}$ is the diagonalizable double extension of $\mathfrak{q}$ by C then $3C^2 = 2C^3 + C$, $\mathfrak{J}$ is not solvable and $\mathfrak{t}\star\mathfrak{t} = \mathfrak{t}$. In the reduced case, y_1 acts diagonalizably on $\mathfrak{J}$ with eigenvalues 1 and $\frac{1}{2}$.*

In Propositions 2.5 and 2.8, we characterize these extensions up to i-isomorphisms, as well as up to isomorphisms and obtain the classification result:

THEOREM 2:

(1) *Let $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ and $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ be nilpotent double extensions of $\mathfrak{q}$ by symmetric maps C and C' respectively. Then there exists a Jordan algebra isomorphism A between $\mathfrak{J}$ and $\mathfrak{J}'$ such that $A(\mathfrak{q}\oplus\mathbb{C}x_1) = \mathfrak{q}\oplus\mathbb{C}x_1'$ if and only if there exist an invertible map $P \in \mathrm{End}(\mathfrak{q})$ and a nonzero $\lambda \in \mathbb{C}$ such that $\lambda C' = PCP^{-1}$ and $P^*PC = C$, where P^* is the adjoint map of P with respect to B. In this case A i-isomorphic then $P \in \mathrm{O}(\mathfrak{q})$.*

(2) *Let* $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ *and* $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x'_1 \oplus \mathbb{C}y'_1)$ *be diagonalizable double extensions of* $\mathfrak{q}$ *by symmetric maps* C *and* C' *respectively. Then* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *are isomorphic if and only if they are i-isomorphic. In this case,* C *and* C' *have the same spectrum.*

In Section 3, we introduce the notion of generalized double extension but with a restricting condition for 2-step nilpotent pseudo-Euclidean Jordan algebras. As a consequence, we obtain in this way the inductive characterization of those algebras (Proposition 3.11):

THEOREM 3:

Let $\mathfrak{J}$ *be a 2-step nilpotent pseudo-Euclidean Jordan algebra. If* $\mathfrak{J}$ *is non-Abelian then it is obtained from an Abelian algebra by a sequence of generalized double extensions.*

To characterize (up to isomorphisms and i-isomorphisms) 2-step nilpotent pseudo-Euclidean Jordan algebras we need to use the concept of a T^*-extension in [Bor97] as follows. Given a complex vector space $\mathfrak{a}$ and a non-degenerate cyclic symmetric bilinear map $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$, define on the vector space $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{a}^*$ the product

$$(x+f)(y+g) = \theta(x,y)$$

then $\mathfrak{J}$ is a 2-step nilpotent pseudo-Euclidean Jordan algebra and it is called a *T^*-extension of* $\mathfrak{a}$ *by* θ (or a T^*-extension, simply). Moreover, we have the following result (Proposition 3.14):

THEOREM 4:

Every reduced 2-step nilpotent pseudo-Euclidean Jordan algebra is i-isomorphic to some T^**-extension.*

Theorem 4 allows us to consider only isomorphic classes and i-isomorphic classes of T^*-extensions to represent all 2-step nilpotent pseudo-Euclidean Jordan algebras. An i-isomorphic and isomorphic characterization of T^*-extensions is given by:

THEOREM 5:

Let $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *be* T^**-extensions of* $\mathfrak{a}$ *by* θ_1 *and* θ_2 *respectively. Then:*

(1) *there exists a Jordan algebra isomorphism between* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *if and only if there exist an isomorphism* A_1 *of* $\mathfrak{a}$ *and an isomorphism* A_2 *of* $\mathfrak{a}^*$ *satisfying:*

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)), \forall x,y \in \mathfrak{a}.$$

(2) *there exists a Jordan algebra i-isomorphism between* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *if and only if there exists an isomorphism* A_1 *of* $\mathfrak{a}$

$$\theta_1(x,y) = \theta_2(A_1(x),A_1(y)) \circ A_1, \forall x,y \in \mathfrak{a}.$$

As a consequence, the classification of i-isomorphic T^*-extensions of $\mathfrak{a}$ is equivalent to the classification of symmetric 3-forms on $\mathfrak{a}$. We detail it in the cases of $\dim(\mathfrak{a}) = 1$ and 2.

In the last Section, we study Novikov algebras. These objects appear in the study of the Hamiltonian condition of an operator in the formal calculus of variations [GD79] and in the classification of Poisson brackets of hydrodynamic type [BN85].

A detailed classification of Novikov algebras up to dimension 3 can be found in [BM01].

An associative algebra is both Lie-admissible and Jordan-admissible. This is not true for Novikov algebras although they are Lie-admissible. Therefore, it is natural to search a condition for a Novikov algebra to become Jordan-admissible. The condition we give here (weaker than associativity) is the following (Proposition 4.17):

THEOREM 6:

A Novikov algebra $\mathfrak{N}$ is Jordan-admissible if it satisfies the condition

$$(x,x,x) = 0, \forall x \in \mathfrak{N}.$$

A corollary of Theorem 6 is that Novikov algebras are not power-associative since there exist Novikov algebras not Jordan-admissible.

Next, we consider symmetric Novikov algebras. A Novikov algebra $\mathfrak{N}$ is called *symmetric* if it is endowed with a non-degenerate associative symmetric bilinear form. In this case, $\mathfrak{N}$ will be associative, its sub-adjacent Lie algebra $\mathfrak{g}(\mathfrak{N})$ is a quadratic 2-step nilpotent Lie algebra [AB10] and the associated Jordan algebra $\mathfrak{J}(\mathfrak{N})$ is pseudo-Euclidean. Therefore, the study of quadratic 2-step nilpotent Lie algebras ([Ova07], [Duo10]) and pseudo-Euclidean Jordan algebras is closely related to symmetric Novikov algebras.

By the results in [ZC07] and [AB10], we have that every symmetric Novikov algebra up to dimension 5 is commutative and a non-commutative example is given in the case of dimension 6. This algebra is 2-step nilpotent. In this paper, we show that every symmetric non-commutative Novikov algebra of dimension 6 is 2-step nilpotent.

As for quadratic Lie algebras and pseudo-Euclidean Jordan algebras, we define the notion of a *reduced* symmetric Novikov algebra. Using this notion, we obtain (Proposition 4.29):

THEOREM 7:

Let $\mathfrak{N}$ be a symmetric non-commutative Novikov algebra. If $\mathfrak{N}$ is reduced then

$$3 \leq \dim(\mathrm{Ann}(\mathfrak{N})) \leq \dim(\mathfrak{N}\mathfrak{N}) \leq \dim(\mathfrak{N}) - 3.$$

In other words, we do not have $\mathfrak{N}\mathfrak{N} = \mathfrak{N}$ in the non-commutative case. Note that this may be true in the commutative case (see Example 4.13). As a result, we obtain the following result for the case of dimension 7 (Proposition 4.32):

THEOREM 8:

Let $\mathfrak{N}$ be a symmetric non-commutative Novikov algebra of dimension 7. If $\mathfrak{N}$ is reduced then there are only two cases:

(1) *$\mathfrak{N}$ is 3-step nilpotent and indecomposable.*
(2) *$\mathfrak{N}$ is decomposable by $\mathfrak{N} = \mathbb{C}x \overset{\perp}{\oplus} \mathfrak{N}_6$, where $x^2 = x$ and $\mathfrak{N}_6$ is a symmetric non-commutative Novikov algebra of dimension 6.*

Finally, we give an example for 3-step nilpotent symmetric Novikov algebras of dimension 7. By the above theorem, it is indecomposable.

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1. PSEUDO-EUCLIDEAN JORDAN ALGEBRAS

Definition 1.1. A (non-associative) algebra $\mathfrak{J}$ over $\mathbb{C}$ is called a (commutative) *Jordan algebra* if its product is commutative and satisfies the following identity (*Jordan identity*):

$$(xy)x^2 = x(yx^2), \forall\, x,y,z \in \mathfrak{J}. \tag{I}$$

For instance, any commutative algebra with an associative product is a Jordan algebra.

Given an algebra A, the *commutator* $[x,y] := xy - yx,\ \forall\ x,y \in A$ measures the commutativity of A. Similarly the *associator* defined by

$$(x,y,z) := (xy)z - x(yz),\ \forall\ x,y,z \in A.$$

measures the associativity of A. In terms of associators, the Jordan identity in a Jordan algebra $\mathfrak{J}$ becomes

$$(x,y,x^2) = 0, \forall\, x,y,z \in \mathfrak{J}. \tag{II}$$

An algebra A is called a power-associative algebra if the subalgebra generated by any element $x \in A$ is associative (see [Sch66] for more details). A Jordan algebra is an example of a power-associative algebra. A power-associative algebra A is called *trace-admissible* if there exists a bilinear form τ on A that satisfies:

(1) $\tau(x,y) = \tau(y,x)$,
(2) $\tau(xy,z) = \tau(x,yz)$,
(3) $\tau(e,e) \neq 0$ for any idempotent e of A,
(4) $\tau(x,y) = 0$ if xy is nilpotent or $xy = 0$.

It is a well-known result that simple (commutative) Jordan algebras are trace-admissible [Alb49]. A similar fact is proved for any *non-commutative* Jordan algebras of characteristic 0 [Sch55]. Recall that non-commutative Jordan algebras are algebras satisfying (I) and the *flexible* condition $(xy)x = x(yx)$ (a weaker condition than commutativity).

A bilinear form B on a Jordan algebra $\mathfrak{J}$ is *associative* if

$$B(xy,z) = B(x,yz), \forall\, x,y,z \in \mathfrak{J}.$$

The following definition is quite natural:

Definition 1.2. Let $\mathfrak{J}$ be a Jordan algebra equipped with an associative symmetric non-degenerate bilinear form B. We say that the pair $(\mathfrak{J},B)$ is a *pseudo-Euclidean Jordan algebra* and B is an *associative scalar product* on $\mathfrak{J}$.

Recall that a real finite-dimensional Jordan algebra $\mathfrak{J}$ with a unit element e (that means, $xe = ex = x,\ \forall x \in \mathfrak{J}$) is called *Euclidean* if there exists an associative inner

product on $\mathfrak{J}$. This is equivalent to say that the associated trace form $\mathrm{Tr}(xy)$ is positive definite, where $\mathrm{Tr}(x)$ is the sum of eigenvalues in the spectral decomposition of $x \in \mathfrak{J}$. To obtain a pseudo-Euclidean Jordan algebra, we replace the base field $\mathbb{R}$ by $\mathbb{C}$ and the inner product by a non-degenerate symmetric bilinear form (considered as a generalized inner product) on $\mathfrak{J}$ keeping its associativity.

Lemma 1.3. *Let $(\mathfrak{J},B)$ be a pseudo-Euclidean Jordan algebra and I be a* **non-degenerate ideal** *of $\mathfrak{J}$, that is, the restriction $B|_{I\times I}$ is non-degenerate. Then $I^\perp$ is also an ideal of $\mathfrak{J}$, $II^\perp = I^\perp I = \{0\}$ and $I \cap I^\perp = \{0\}$.*

Proof. Let $x \in I^\perp, y \in \mathfrak{J}$, one has $B(xy,I) = B(x,yI) = 0$ then $xy \in I^\perp$ and $I^\perp$ is an ideal.

If $x \in I^\perp$ such that $B(x,I^\perp) = 0$ then $x \in I$ and $B(x,I) = 0$. Since I is non-degenerate then $x = 0$. That implies that $I^\perp$ is non-degenerate.

Since $B(II^\perp,\mathfrak{J}) = B(I,I^\perp\mathfrak{J}) = 0$ then $II^\perp = I^\perp I = \{0\}$.

If $x \in I \cap I^\perp$ then $B(x,I) = 0$. Since I non-degenerate, then $x = 0$. □

By the proof of above Lemma, given a non-degenerate subspace W of $\mathfrak{J}$ then $W^\perp$ is also non-degenerate and $\mathfrak{J} = W \oplus W^\perp$. In this case, we use the notation:

$$\mathfrak{J} = W \overset{\perp}{\oplus} W^\perp.$$

Remark 1.4. A pseudo-Euclidean Jordan algebra does not necessarily have a unit element. However if that is the case, this unit element is certainly unique. A Jordan algebra with unit element is called a *unital* Jordan algebra. If $\mathfrak{J}$ is not a unital Jordan algebra, we can extend $\mathfrak{J}$ to a unital Jordan algebra $\overline{\mathfrak{J}} = \mathbb{C}e \oplus \mathfrak{J}$ by the product

$$(\lambda e + x) \star (\mu e + y) = \lambda\mu e + \lambda y + \mu x + xy.$$

More particularly, $e \star e = e$, $e \star x = x \star e = x$ and $x \star y = xy$. In this case, we say $\overline{\mathfrak{J}}$ the *unital extension* of $\mathfrak{J}$.

Proposition 1.5. *If $(\mathfrak{J},B)$ is unital then there is a decomposition:*

$$\mathfrak{J} = \mathfrak{J}_1 \overset{\perp}{\oplus} \dots \overset{\perp}{\oplus} \mathfrak{J}_k,$$

where $\mathfrak{J}_i$, $i = 1,\dots,k$ are unital and indecomposable ideals.

Proof. The assertion is obvious if $\mathfrak{J}$ is indecomposable. Assume that $\mathfrak{J}$ is decomposable, that is, $\mathfrak{J} = I \oplus I'$ with I, $I' \neq \{0\}$ proper ideals of $\mathfrak{J}$ such that I is non-degenerate. By the above Lemma, $I' = I^\perp$ and we write $\mathfrak{J} = I \overset{\perp}{\oplus} I^\perp$. Assume that $\mathfrak{J}$ has the unit element e. If $e \in I$ then for x a nonzero element in $I^\perp$, we have $ex = x \in I^\perp$. This is a contradiction. This happens similarly if $e \in I^\perp$. Therefore, $e = e_1 + e_2$ where $e_1 \in I$ and $e_2 \in I^\perp$ are nonzero vectors. For all $x \in I$, one has:

$$x = ex = (e_1 + e_2)x = e_1 x = x e_1.$$

It implies that e_1 is the unit element of I. Similarly, e_2 is also the unit element of $I^\perp$. Since the dimension of $\mathfrak{J}$ is finite then by induction, one has the result. □

Example 1.6. Let us recall an example in Chapter II of [FK94]: consider $\mathfrak{q}$ a vector space over $\mathbb{C}$ and $B : \mathfrak{q} \times \mathfrak{q} \to \mathbb{C}$ a symmetric bilinear form. Define the product below on the vector space $\mathfrak{J} = \mathbb{C}e \oplus \mathfrak{q}$:

$$(\lambda e + u)(\mu e + v) := (\lambda \mu + B(u,v))e + \lambda v + \mu u,$$

for all $\lambda, \mu \in \mathbb{C}, u, v \in \mathfrak{q}$. In particular, $e^2 = e$, $ue = eu = u$ and $uv = B(u,v)e$. This product makes $\mathfrak{J}$ a Jordan algebra.

Now, we add the condition that B is non-degenerate and define a bilinear form $B_{\mathfrak{J}}$ on $\mathfrak{J}$ by:

$$B_{\mathfrak{J}}(e,e) = 1,\ B_{\mathfrak{J}}(e,\mathfrak{q}) = B_{\mathfrak{J}}(\mathfrak{q},e) = 0 \text{ and } B_{\mathfrak{J}}|_{\mathfrak{q}\times\mathfrak{q}} = B.$$

Then $B_{\mathfrak{J}}$ is associative and non-degenerate and $\mathfrak{J}$ becomes a pseudo-Euclidean Jordan algebra with unit element e.

Example 1.7. Let us slightly change Example 1.6 by setting

$$\mathfrak{J}' := \mathbb{C}e \oplus \mathfrak{q} \oplus \mathbb{C}f.$$

Define the product of $\mathfrak{J}'$ as follows:

$$e^2 = e,\ ue = eu = u,\ ef = fe = f,\ uv = B(u,v)f \text{ and } uf = fu = ff = 0,$$

for all $u, v \in \mathfrak{q}$. It is easy to see that $\mathfrak{J}'$ is the unital extension of the Jordan algebra $\mathfrak{J} = \mathfrak{q} \oplus \mathbb{C}f$, where the product on $\mathfrak{J}$ is defined by:

$$uv = B(u,v)f,\ uf = fu = 0, \forall u, v \in \mathfrak{q}.$$

Moreover, $\mathfrak{J}'$ is a pseudo-Euclidean Jordan algebra with the bilinear form $B_{\mathfrak{J}'}$ defined by:

$$B_{\mathfrak{J}'}\left(\lambda e + u + \lambda' f, \mu e + v + \mu' f\right) = \lambda \mu' + \lambda' \mu + B(u,v).$$

We will meet this algebra again in the next Section.

Recall the definition of a representation of a Jordan algebra:

Definition 1.8. A *Jacobson representation* (or simply, a *representation*) of a Jordan algebra $\mathfrak{J}$ on a vector space V is a linear map $\mathfrak{J} \to \mathrm{End}(V)$, $x \mapsto S_x$ satisfying for all $x, y, z \in \mathfrak{J}$,

(1) $[S_x, S_{yz}] + [S_y, S_{zx}] + [S_z, S_{xy}] = 0,$
(2) $S_x S_y S_z + S_z S_y S_x + S_{(xz)y} = S_x S_{yz} + S_y S_{zx} + S_z S_{xy}.$

Remark 1.9. An equivalent definition of a representation of $\mathfrak{J}$ can be found for instance in [BB], as a necessary and sufficient condition for the vector space $\mathfrak{J}_1 = \mathfrak{J} \oplus V$ equipped with the product:

$$(x+u)(y+v) = xy + S_x(v) + S_y(u),\ \forall x, y \in \mathfrak{J}, u, v \in V$$

to be a Jordan algebra. In this case, Jacobson's definition is different from the usual definition of representation, that is, as a homomorphism from $\mathfrak{J}$ into the Jordan algebra of linear maps.

For $x \in \mathfrak{J}$, let $R_x \in \mathrm{End}(\mathfrak{J})$ be the endomorphism of $\mathfrak{J}$ defined by:

$$R_x(y) = xy = yx, \forall\, y \in \mathfrak{J}.$$

Then the Jordan identity is equivalent to $[R_x, R_{x^2}] = 0, \forall\, x \in \mathfrak{J}$ where $[\cdot,\cdot]$ denotes the Lie bracket on $\mathrm{End}(\mathfrak{J})$. The linear maps

$$R : \mathfrak{J} \to \mathrm{End}(\mathfrak{J}) \ \text{ with } R(x) := R_x$$

$$\text{and } R^* : \mathfrak{J} \to \mathrm{End}(\mathfrak{J}^*) \ \text{ with } R^*(x)(f) = f \circ R_x, \forall\, x \in \mathfrak{J}, f \in \mathfrak{J}^*,$$

are called respectively the *adjoint representation* and the *coadjoint representation* of $\mathfrak{J}$. It is easy to check that they are indeed representations of $\mathfrak{J}$. Recall that there exists a natural non-degenerate bilinear from $\langle \cdot,\cdot \rangle$ on $\mathfrak{J} \oplus \mathfrak{J}^*$ defined by $\langle x, f \rangle := f(x),\ \forall x \in \mathfrak{J},\ f \in \mathfrak{J}^*$. For all $x, y \in \mathfrak{J}, f \in \mathfrak{J}^*$, one has:

$$f(xy) = \langle xy, f \rangle = \langle R_x(y), f \rangle = \langle y, R^*_x(f) \rangle.$$

That means that R^*_x is the adjoint map of R_x with respect to the bilinear form $\langle \cdot,\cdot \rangle$.

The following proposition gives a characterization of pseudo-Euclidean Jordan algebras. A proof can be found in [BB], Proposition 2.1 or [Bor97], Proposition 2.4.

Proposition 1.10. *Let $\mathfrak{J}$ be a Jordan algebra. Then $\mathfrak{J}$ is pseudo-Euclidean if, and only if, its adjoint representation and coadjoint representation are equivalent.*

We will need some special subspaces of an arbitrary algebra $\mathfrak{J}$:

Definition 1.11. Let $\mathfrak{J}$ be an algebra.

(1) The subspace

$$(\mathfrak{J},\mathfrak{J},\mathfrak{J}) := \mathrm{span}\{(x,y,z) \mid x,y,z \in \mathfrak{J}\}$$

is the *associator* of $\mathfrak{J}$.

(2) The subspaces

$$\mathrm{LAnn}(\mathfrak{J}) := \{x \in \mathfrak{J} \mid x\mathfrak{J} = 0\},$$
$$\mathrm{RAnn}(\mathfrak{J}) := \{x \in \mathfrak{J} \mid \mathfrak{J}x = 0\} \text{ and}$$
$$\mathrm{Ann}(\mathfrak{J}) := \{x \in \mathfrak{J} \mid x\mathfrak{J} = \mathfrak{J}x = 0\}$$

are respectively the *left-annulator*, the *right-annulator* and the *annulator* of $\mathfrak{J}$. Certainly, if $\mathfrak{J}$ is commutative then these subspaces coincide.

(3) The subspace

$$N(\mathfrak{J}) := \{x \in \mathfrak{J} \mid (x,y,z) = (y,x,z) = (y,z,x) = 0, \forall y,z \in \mathfrak{J}\}$$

is the *nucleus* of $\mathfrak{J}$.

The proof of the Proposition below is straightforward and we omit it.

Proposition 1.12. *If $(\mathfrak{J}, B)$ is a pseudo-Euclidean Jordan algebra then*

(1) *the nucleus $N(\mathfrak{J})$ coincide with the* **center** $\mathsf{Z}(\mathfrak{J})$ *of $\mathfrak{J}$ where $\mathsf{Z}(\mathfrak{J}) = \{x \in N(\mathfrak{J}) \mid xy = yx, \forall y \in \mathfrak{J}\}$, that is, the set of all elements x that commute and associate with all elements of $\mathfrak{J}$. Therefore*

$$N(\mathfrak{J}) = \mathsf{Z}(\mathfrak{J}) = \{x \in \mathfrak{J} \mid (x,y,z) = 0, \forall y,z \in \mathfrak{J}\}.$$

(2) $\mathsf{Z}(\mathfrak{J})^{\perp} = (\mathfrak{J},\mathfrak{J},\mathfrak{J})$.
(3) $(\mathrm{Ann}(\mathfrak{J}))^{\perp} = \mathfrak{J}^2$.

Just as in [DPU] where we have defined reduced quadratic Lie algebras, we can define here:

Definition 1.13. A pseudo-Euclidean Jordan algebra $(\mathfrak{J},B)$ is *reduced* if

(1) $\mathfrak{J} \neq \{0\}$,
(2) $\mathrm{Ann}(\mathfrak{J})$ is totally isotropic, that means $B(x,y) = 0$ for all $x,y \in \mathrm{Ann}(\mathfrak{J})$.

Proposition 1.14. *Let $\mathfrak{J}$ be non-Abelian pseudo-Euclidean Jordan algebra. Then $\mathfrak{J} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$, where $\mathfrak{z} \subset \mathrm{Ann}(\mathfrak{J})$ and $\mathfrak{l}$ is reduced.*

Proof. The proof is completely similar to Proposition 6.7 in [PU07]. Let $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}^2$, $\mathfrak{z}$ is a complementary subspace of $\mathfrak{z}_0$ in $\mathrm{Ann}(\mathfrak{J})$ and $\mathfrak{l} = \mathfrak{z}^{\perp}$. If x is an element in $\mathfrak{z}$ such that $B(x,\mathfrak{z}) = 0$ then $B(x,\mathfrak{J}^2) = 0$ since $\mathrm{Ann}(\mathfrak{J}) = (\mathfrak{J}^2)^{\perp}$. As a consequence, $B(x,\mathfrak{z}_0) = 0$ and therefore $B(x,\mathrm{Ann}(\mathfrak{J})) = 0$. That implies $x \in \mathfrak{J}^2$. Hence, $x = 0$ and the restriction of B to $\mathfrak{z}$ is non-degenerate. Moreover, $\mathfrak{z}$ is an ideal then by Lemma 1.3, the restriction of B to $\mathfrak{l}$ is also a non-degenerate and that $\mathfrak{z} \cap \mathfrak{l} = \{0\}$.

Since $\mathfrak{J}$ is non-Abelian then $\mathfrak{l}$ is non-Abelian and $\mathfrak{l}^2 = \mathfrak{J}^2$. Moreover, $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{l})$ and the result follows. □

Next, we will define some extensions of a Jordan algebra and introduce the notion of a *double extension* of a pseudo-Euclidean Jordan algebra [BB].

Definition 1.15. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be Jordan algebras and $\pi : \mathfrak{J}_1 \to \mathrm{End}(\mathfrak{J}_2)$ be a representation of $\mathfrak{J}_1$ on $\mathfrak{J}_2$. We call π an *admissible representation* if it satisfies the following conditions:

(1) $\pi(x^2)(yy') + 2(\pi(x)y')(\pi(x)y) + (\pi(x)y')y^2 + 2(yy')(\pi(x)y)$
$= 2\pi(x)(y'(\pi(x)y)) + \pi(x)(y'y^2) + (\pi(x^2)y')y + 2(y'(\pi(x)y))y$,
(2) $(\pi(x)y)y^2 = (\pi(x)y^2)y$,
(3) $\pi(xx')y^2 + 2(\pi(x')y)(\pi(x)y) = \pi(x)\pi(x')y^2 + 2(\pi(x')\pi(x)y)y$,

for all $x,x' \in \mathfrak{J}_1, y,y' \in \mathfrak{J}_2$. In this case, the vector space $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ with the product defined by:

$$(x+y)(x'+y') = xx' + \pi(x)y' + \pi(x')y + yy',\ \forall x,x' \in \mathfrak{J}_1, y,y' \in \mathfrak{J}_2$$

becomes a Jordan algebra.

Definition 1.16. Let $(\mathfrak{J},B)$ be a pseudo-Euclidean Jordan algebra and C be an endomorphism of $\mathfrak{J}$. We say that C is *symmetric* if

$$B(C(x),y) = B(x,C(y)), \forall x,y \in \mathfrak{J}.$$

Denote by $\mathrm{End}_s(\mathfrak{J})$ the space of symmetric endomorphisms of $\mathfrak{J}$.

The definition below was introduced in [BB], Theorem 3.8.

Definition 1.17. Let $(\mathfrak{J}_1, B_1)$ be a pseudo-Euclidean Jordan algebra, $\mathfrak{J}_2$ be an arbitrary Jordan algebra and $\pi : \mathfrak{J}_2 \to \mathrm{End}_s(\mathfrak{J}_1)$ be an admissible representation. Define a symmetric bilinear map $\varphi : \mathfrak{J}_1 \times \mathfrak{J}_1 \to \mathfrak{J}_2^*$ by: $\varphi(y,y')(x) = B_1(\pi(x)y,y'), \forall x \in \mathfrak{J}_2, y,y' \in \mathfrak{J}_1$. Consider the vector space

$$\overline{\mathfrak{J}} = \mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*$$

endowed with the product:

$$(x+y+f)(x'+y'+f') = xx' + yy' + \pi(x)y' + \pi(x')y + f' \circ R_x + f \circ R_{x'} + \varphi(y,y')$$

for all $x,x' \in \mathfrak{J}_2, y,y' \in \mathfrak{J}_1, f, f' \in \mathfrak{J}_2^*$. Then $\overline{\mathfrak{J}}$ is a Jordan algebra. Moreover, define a bilinear form B on $\overline{\mathfrak{J}}$ by:

$$B(x+y+f, x'+y'+f') = B_1(y,y') + f(x') + f'(x), \forall x,x' \in \mathfrak{J}_2, y,y' \in \mathfrak{J}_1, f, f' \in \mathfrak{J}_2^*.$$

Then $\overline{\mathfrak{J}}$ is a **pseudo-Euclidean Jordan algebra**. The Jordan algebra $(\overline{\mathfrak{J}}, B)$ is called the *double extension* of $\mathfrak{J}_1$ by $\mathfrak{J}_2$ by means of π.

Remark 1.18. If γ is an associative bilinear form (not necessarily non-degenerate) on $\mathfrak{J}_2$ then $\overline{\mathfrak{J}}$ is again pseudo-Euclidean thanks to the bilinear form

$$B_\gamma(x+y+f, x'+y'+f') = \gamma(x,x') + B_1(y,y') + f(x') + f'(x)$$

for all $x, x' \in \mathfrak{J}_2, y, y' \in \mathfrak{J}_1, f, f' \in \mathfrak{J}_2^*$.

2. Jordanian double extension of a quadratic vector space

Let $\mathbb{C}c$ be a one-dimensional Jordan algebra. If $c^2 \neq 0$ then $c^2 = \lambda c$ for some nonzero $\lambda \in \mathbb{C}$. Replace $c := \frac{1}{\lambda}c$, we obtain $c^2 = c$. Therefore, there exist only two one-dimensional Jordan algebras: one Abelian and one simple. Next, we will study double extensions of a quadratic vector space by these algebras.

Let us start with $(\mathfrak{q}, B_\mathfrak{q})$ a **quadratic vector space**, that is, $B_\mathfrak{q}$ is a non-degenerate symmetric bilinear form on the vector space $\mathfrak{q}$. We consider $(\mathfrak{t} = \mathrm{span}\{x_1,y_1\}, B_\mathfrak{t})$ a 2-dimensional quadratic vector space with the bilinear form $B_\mathfrak{t}$ defined by

$$B_\mathfrak{t}(x_1,x_1) = B_\mathfrak{t}(y_1,y_1) = 0, \; B_\mathfrak{t}(x_1,y_1) = 1.$$

Let $C : \mathfrak{q} \to \mathfrak{q}$ be a nonzero symmetric map and consider the vector space

$$\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$$

equipped with a product defined by

$$\begin{aligned}(x+\lambda x_1 + \mu y_1) \quad & (y + \lambda' x_1 + \mu' y_1) := \\ & \mu C(y) + \mu' C(x) + B_\mathfrak{q}(C(x),y)x_1 + \varepsilon\left(\left(\lambda\mu' + \lambda'\mu\right)x_1 + \mu\mu' y_1\right),\end{aligned}$$

$\varepsilon \in \{0,1\}$, for all $x,y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$.

Proposition 2.1. *Keep the notation just above.*

(1) *Assume $\varepsilon = 0$. Then $\mathfrak{J}$ is a Jordan algebra if, and only if, $C^3 = 0$. In this case, we call $\mathfrak{J}$ a* **nilpotent double extension** *of $\mathfrak{q}$ by C.*
(2) *Assume $\varepsilon = 1$. Then $\mathfrak{J}$ is a Jordan algebra if, and only if, $3C^2 = 2C^3 + C$. Moreover, $\mathfrak{J}$ is pseudo-Euclidean with the bilinear form $B = B_\mathfrak{q} + B_\mathfrak{t}$. In this case, we call $\mathfrak{J}$ a* **diagonalizable double extension** *of $\mathfrak{q}$ by C.*

Proof.

(1) Let $x, y \in \mathfrak{q}$, λ, μ, λ', $\mu' \in \mathbb{C}$. One has

$$((x+\lambda x_1+\mu y_1)(y+\lambda' x_1+\mu' y_1))(x+\lambda x_1+\mu y_1)^2 = 2\mu B_{\mathfrak{q}}(C^2(\mu y+\mu' x),C(x))x_1$$

and

$$(x+\lambda x_1+\mu y_1)((y+\lambda' x_1+\mu' y_1)(x+\lambda x_1+\mu y_1)^2) = 2\mu^2\mu' C^3(x) + 2\mu\mu' B_{\mathfrak{q}}(C(x),C^2(x))x_1.$$

Therefore, $\mathfrak{J}$ is a Jordan algebra if and only if $C^3 = 0$.

(2) The result is achieved by checking directly the equality (I) for $\mathfrak{J}$.

□

2.1. Nilpotent double extensions.

Consider $\mathfrak{J}_1 := \mathfrak{q}$ an Abelian algebra, $\mathfrak{J}_2 := \mathbb{C}y_1$ the nilpotent one-dimensional Jordan algebra, $\pi(y_1) := C$ and identify $\mathfrak{J}_2^*$ with $\mathbb{C}x_1$. Then by Definition 1.17, $\mathfrak{J} = \mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*$ is a pseudo-Euclidean Jordan algebra with a bilinear form B given by $B := B_{\mathfrak{q}} + B_{\mathfrak{t}}$. In this case, C obviously satisfies the condition $C^3 = 0$.

An immediate corollary of the definition is:

Corollary 2.2. *If* $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ *is the nilpotent double extension of* $\mathfrak{q}$ *by* C *then*

$$y_1 x = C(x), xy = B(C(x),y)x_1 \text{ and } y_1 y_1 = x_1 \mathfrak{J} = 0, \forall x \in \mathfrak{q}.$$

As a consequence, $\mathfrak{J}^2 = \mathrm{Im}(C) \oplus \mathbb{C}x_1$ *and* $Ann(\mathfrak{J}) = \ker(C) \oplus \mathbb{C}x_1$.

Remark 2.3. In this case, $\mathfrak{J}$ is k-step nilpotent, $k \le 3$ since $R_x^k(\mathfrak{J}) \subset \mathrm{Im}(C^k) \oplus \mathbb{C}x_1$.

Definition 2.4. Let (V,B) and (V',B') be two quadratic vector spaces. An *isometry* is a bijective map $A : V \to V'$ that satisfies

$$B'(A(v),A(w)) = B(v,w),\ \forall u,v \in V.$$

The group of isometries of V is denoted by $\mathrm{O}(V,B)$ (or simply $\mathrm{O}(V)$). In the case $(\mathfrak{J},B)$ and $(\mathfrak{J}',B')$ are pseudo-Euclidean Jordan algebras, if there exists a Jordan algebra isomorphism A between $\mathfrak{J}$ and $\mathfrak{J}'$ such that it is also an isometry then we say $\mathfrak{J}$, $\mathfrak{J}'$ are *i-isomorphic* and A is an *i-isomorphism*.

Proposition 2.5. *Let* $(\mathfrak{q},B)$ *be a quadratic vector space. Let* $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ *and* $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ *be nilpotent double extensions of* $\mathfrak{q}$*, by symmetric maps* C *and* C' *respectively. Then:*

(1) *there exists a Jordan algebra isomorphism* A *between* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *such that* $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$ *if, and only if, there exists an invertible map* $P \in \mathrm{End}(\mathfrak{q})$ *and a nonzero* $\lambda \in \mathbb{C}$ *such that* $\lambda C' = PCP^{-1}$ *and* $P^*PC = C$*, where* P^* *is the adjoint map of* P *with respect to* B.

(2) *there exists a Jordan algebra i-isomorphism* A *between* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *such that* $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$ *if, and only if, there exists a nonzero* $\lambda \in \mathbb{C}$ *such that* C *and* $\lambda C'$ *are conjugate by an isometry* $P \in \mathrm{O}(\mathfrak{q})$.

Proof.

(1) Assume $A : \mathfrak{J} \to \mathfrak{J}'$ be an isomorphism such that $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x'_1$. Since $x_1 \in \mathfrak{J}^2$, then there exist $x, y \in \mathfrak{J}$ such that $xy = x_1$ (by Proposition 2.1). Therefore $A(x_1) = A(x)A(y) \in (\mathfrak{q} \oplus \mathbb{C}x'_1)(\mathfrak{q} \oplus \mathbb{C}x'_1) = \mathbb{C}x'_1$. That means $A(x_1) = \mu x'_1$ for some nonzero $\mu \in \mathbb{C}$. Write $A|_{\mathfrak{q}} = P + \beta \otimes x'_1$ with $P \in \mathrm{End}(\mathfrak{q})$ and $\beta \in \mathfrak{q}^*$. If $x \in \ker(P)$ then $A\left(x - \dfrac{1}{\mu}\beta(x)x_1\right) = 0$, so $x = 0$ and therefore, P is invertible. For all $x, y \in \mathfrak{q}$, one has

$$\mu B(C(x), y)x'_1 = A(xy) = A(x)A(y) = B(C'(P(x)), P(y))x'_1.$$

So we obtain $P^*C'P = \mu C$. Assume $A(y_1) = y + \delta x'_1 + \lambda y'_1$, with $y \in \mathfrak{q}$. For all $x \in \mathfrak{q}$, one has

$$P(C(x)) + \beta(C(x))x'_1 = A(y_1 x) = A(y_1)A(x) = \lambda C'(P(x)) + B(C'(y), P(x))x'_1.$$

Therefore, $\lambda C' = PCP^{-1}$. Combine with $P^*C'P = \mu C$ to get $P^*PC = \lambda\mu C$. Replace P by $\dfrac{1}{(\mu\lambda)^{\frac{1}{2}}}P$ to obtain $\lambda C' = PCP^{-1}$ and $P^*PC = C$.

Conversely, define $A : \mathfrak{J} \to \mathfrak{J}'$ by $A(y_1) = \lambda y'_1$, $A(x) = P(x), \forall x \in \mathfrak{q}$ and $A(x_1) = \frac{1}{\lambda}x'_1$ then it is easy to check A is an isomorphism.

(2) If $A : \mathfrak{J} \to \mathfrak{J}'$ is an i-isomorphic then the isomorphism P in the proof of (1) is also an isometry. Hence $P \in \mathrm{O}(\mathfrak{q})$. Conversely, define A as in (1) then it is obvious that A is an i-isomorphism.

□

Proposition 2.6. *Let $(\mathfrak{q}, B)$ be a quadratic vector space, $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$, $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x'_1 \oplus \mathbb{C}y'_1)$ be nilpotent double extensions of $\mathfrak{q}$, by symmetric maps C and C' respectively. Assume that* $\mathrm{rank}(C') \geq 3$. *Let A be an isomorphism between $\mathfrak{J}$ and $\mathfrak{J}'$. Then $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x'_1$.*

Proof. We assume that there is $x \in \mathfrak{q}$ such that $A(x) = y + \beta x'_1 + \gamma y'_1$, where $y \in \mathfrak{q}, \beta, \gamma \in \mathbb{C}, \gamma \neq 0$. Then for all $q \in \mathfrak{q}$ and $\lambda \in \mathbb{C}$, we have

$$A(x)(q + \lambda x'_1) = \gamma C'(q) + B(C'(y), q)x'_1.$$

Therefore, $\dim(A(x)(\mathfrak{q} \oplus \mathbb{C}x'_1)) \geq 3$. But A is an isomorphism, hence

$$A(x)(\mathfrak{q} \oplus \mathbb{C}x'_1) \subset A(xA^{-1}(\mathfrak{q} \oplus \mathbb{C}x'_1)) \subset A(x(\mathfrak{q} \oplus \mathbb{C}x_1 \oplus \mathbb{C}y_1)) \subset A(\mathbb{C}C(x) \oplus \mathbb{C}x_1).$$

This is a contradiction. Hence $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x'_1$. □

2.2. Diagonalizable double extensions.

Lemma 2.7. *Let $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ be the diagonalizable double extension of $\mathfrak{q}$ by C. Then*

$$y_1 y_1 = y_1, y_1 x_1 = x_1, y_1 x = C(x), xy = B(C(x), y)x_1 \textit{ and } x_1 x = x_1 x_1 = 0, \forall x \in \mathfrak{q}.$$

Note that $x_1 \notin \mathrm{Ann}(\mathfrak{J})$. Let $x \in \mathfrak{q}$. Then $x \in \mathrm{Ann}(\mathfrak{J})$ if and only if $x \in \ker(C)$. Moreover, $\mathfrak{J}^2 = \mathrm{Im}(C) \oplus (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$. Therefore $\mathfrak{J}$ is reduced if, and only if, $\ker(C) \subset \mathrm{Im}(C)$.

Let $x \in \mathrm{Im}(C)$. Then there exists $y \in \mathfrak{q}$ such that $x = C(y)$. Since $3C^2 = 2C^3 + C$, one has $3C(x) - 2C^2(x) = x$. Therefore, if $\mathfrak{J}$ is reduced then $\ker(C) = \{0\}$ and C is invertible. That implies that $3C - 2C^2 = \mathrm{Id}$ and we have the following proposition:

Proposition 2.8. *Let $(\mathfrak{q}, B)$ be a quadratic vector space. Let $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ and $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ be diagonalizable double extensions of $\mathfrak{q}$, by invertible maps C and C' respectively. Then there exists a Jordan algebra isomorphism A between $\mathfrak{J}$ and $\mathfrak{J}'$ if and only if there exists an isometry P such that $C' = PCP^{-1}$. In this case, $\mathfrak{J}$ and $\mathfrak{J}'$ are also i-isomorphic.*

Proof. Assume $\mathfrak{J}$ and $\mathfrak{J}'$ isomorphic by A. Firstly, we will show that $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$. Indeed, if $A(x_1) = y + \beta x_1' + \gamma y_1'$, where $y \in \mathfrak{q}, \beta, \gamma \in \mathbb{C}$, then

$$0 = A(x_1 x_1) = A(x_1)A(x_1) = 2\gamma C'(y) + (2\beta\gamma + B(C'(y), y))x_1' + \gamma^2 y_1'.$$

Therefore, $\gamma = 0$. Similarly, if there exists $x \in \mathfrak{q}$ such that $A(x) = z + \alpha x_1' + \delta y_1'$, where $z \in \mathfrak{q}, \alpha, \delta \in \mathbb{C}$. Then

$$B(C(x), x)A(x_1) = A(xx) = A(x)A(x) = 2\delta C'(y) + (2\alpha\delta + B(C'(z), z))x_1' + \delta^2 y_1'.$$

That implies $\delta = 0$ and $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$.

The rest of the proof follows exactly the proof of Proposition 2.5, one has $A(x_1) = \mu x_1'$ for some nonzero $\mu \in \mathbb{C}$ and there is an isomorphism P of $\mathfrak{q}$ such that $A|_{\mathfrak{q}} = P + \beta \otimes x_1'$, where $\beta \in \mathfrak{q}^*$. Similarly as in the proof of Proposition 2.5, one also has $P^* C' P = \mu C$, where P^* is the adjoint map of P with respect to B. Assume $A(y_1) = \lambda y_1' + y + \delta x_1$. Since $A(y_1)A(y_1) = A(y_1)$, one has $\lambda = 1$ and therefore $C' = PCP^{-1}$. Replace $P := \dfrac{1}{(\mu)^{\frac{1}{2}}} P$ to get $P^* P C = C$. However, since C is invertible then $P^* P = \mathrm{Id}$. That means that P is an isometry of $\mathfrak{q}$.

Conversely, define $A : \mathfrak{J} \to \mathfrak{J}'$ by $A(x_1) = x_1'$, $A(y_1) = y_1'$ and $A(x) = P(x), \forall x \in \mathfrak{q}$ then A is an i-isomorphism. $\square$

An invertible symmetric endomorphism of $\mathfrak{q}$ satisfying $3C - 2C^2 = \mathrm{Id}$ is diagonalizable by an orthogonal basis of eigenvectors with eigenvalues 1 and $\frac{1}{2}$ (see Appendix). Therefore, we have the following corollary:

Corollary 2.9. *Let $(\mathfrak{q}, B)$ be a quadratic vector space. Let $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ and $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ be diagonalizable double extensions of $\mathfrak{q}$, by invertible maps C and C' respectively. Then $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic if , and only if, C and C' have same spectrum.*

Example 2.10. Let $\mathbb{C}x$ be one-dimensional Abelian algebra, $\mathfrak{J} = \mathbb{C}x \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ and $\mathfrak{J}' = \mathbb{C}x \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ be diagonalizable double extensions of $\mathbb{C}x$ by

$C = \mathrm{Id}$ and $C' = \frac{1}{2}\mathrm{Id}$. In particular, the product on $\mathfrak{J}$ and $\mathfrak{J}'$ are defined by:

$$y_1^2 = y_1, y_1x = x, y_1x_1 = x_1, x^2 = x_1;$$
$$(y_1')^2 = y_1', y_1'x = \frac{1}{2}x, y_1x_1 = x_1, x^2 = \frac{1}{2}x_1.$$

Then $\mathfrak{J}$ and $\mathfrak{J}'$ are not isomorphic. Moreover, $\mathfrak{J}'$ has no unit element.

Remark 2.11. The i-isomorphic and isomorphic notions are not coincident in general. For example, the Jordan algebras $\mathfrak{J} = \mathbb{C}e$ with $e^2 = e$, $B(e,e) = 1$ and $\mathfrak{J}' = \mathbb{C}e'$ with $e'e' = e'$, $B(e',e') = a \neq 1$ are isomorphic but not i-isomorphic.

3. Pseudo-Euclidean 2-step nilpotent Jordan algebras

Quadratic 2-step nilpotent Lie algebras are characterized up to isometric isomorphisms and up to isomorphisms in [Ova07]. There is a similar natural property in the case of pseudo-Euclidean 2-step nilpotent Jordan algebras.

3.1. **2-step nilpotent Jordan algebras.**

Let us redefine 2-step nilpotent Jordan algebras in a more convenient way:

Definition 3.1. An algebra $\mathfrak{J}$ over $\mathbb{C}$ with a product $(x,y) \mapsto xy$ is called *2-step nilpotent Jordan algebra* if it satisfies $xy = yx$ and $(xy)z = 0$ for all $x,y,z \in \mathfrak{J}$. Sometimes, we use **2SN-Jordan Algebra** as an abbreviation.

The method of double extension is a fundamental tool used in describing algebras that are endowed with an associative non-degenerate bilinear form. This method is based on two principal notions: central extension and semi-direct sum of two algebras. In the next part, we will recall some definitions given in Section 3 of [BB] but with a restricting condition for pseudo-Euclidean 2-step nilpotent Jordan algebras.

Proposition 3.2. *Let $\mathfrak{J}$ be a 2SN-Jordan algebra, V be a vector space, $\varphi : \mathfrak{J}\times\mathfrak{J} \to V$ be a bilinear map and $\pi : \mathfrak{J} \to \mathrm{End}(V)$ be a representation. Let*

$$\overline{\mathfrak{J}} = \mathfrak{J} \oplus V$$

equipped with the following product:

$$(x+u)(y+v) = xy + \pi(x)(v) + \pi(y)(u) + \varphi(x,y), \forall x,y \in \mathfrak{J}, u,v \in V.$$

Then $\overline{\mathfrak{J}}$ is a 2SN-Jordan algebra if and only if for all $x,y,z \in \mathfrak{J}$:

(1) *φ is symmetric and $\varphi(xy,z) + \pi(z)(\varphi(x,y)) = 0$,*
(2) *$\pi(xy) = \pi(x)\pi(y) = 0$.*

Definition 3.3. If π is the trivial representation in Proposition 3.2, the Jordan algebra $\overline{\mathfrak{J}}$ is called the *2SN-central extension* of $\mathfrak{J}$ by V (by means of φ).

Remark that in a 2SN-central extension $\overline{\mathfrak{J}}$, the annulator $\mathrm{Ann}(\overline{\mathfrak{J}})$ contains the vector space V.

Proposition 3.4. *Let $\mathfrak{J}$ be a 2SN-Jordan algebra. Then $\mathfrak{J}$ is a 2SN-central extension of an Abelian algebra.*

Proof. Set $\mathfrak{h} := \mathfrak{J}/\mathfrak{J}^2$ and $V := \mathfrak{J}^2$. Define $\varphi : \mathfrak{h}\times\mathfrak{h} \to V$ by $\varphi(p(x), p(y)) = xy, \forall x, y \in \mathfrak{J}$, where $p : \mathfrak{J} \to \mathfrak{h}$ is the canonical projection. Then $\mathfrak{h}$ is an Abelian algebra and $\mathfrak{J} \cong \mathfrak{h}\oplus V$ is the 2SN-central extension of $\mathfrak{h}$ by means of φ. □

Remark 3.5. It is easy to see that if $\mathfrak{J}$ is a 2SN-Jordan algebra, then the coadjoint representation R^* of $\mathfrak{J}$ satisfies the condition on π in Proposition 3.2 (2). For a trivial φ, we conclude that $\mathfrak{J}\oplus\mathfrak{J}^*$ is also a 2SN-Jordan algebra with respect to the coadjoint representation.

Definition 3.6. Let $\mathfrak{J}$ be a 2SN-Jordan algebra, V and W be two vector spaces. Let $\pi : \mathfrak{J} \to \mathrm{End}(V)$ and $\rho : \mathfrak{J} \to \mathrm{End}(W)$ be representations of $\mathfrak{J}$. The *direct sum* $\pi\oplus\rho : \mathfrak{J} \to \mathrm{End}(V\oplus W)$ of π and ρ is defined by

$$(\pi\oplus\rho)(x)(v+w) = \pi(x)(v)+\rho(x)(w), \forall x \in \mathfrak{J}, v \in V, w \in W.$$

Proposition 3.7. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be 2SN-Jordan algebras and $\pi : \mathfrak{J}_1 \to \mathrm{End}(\mathfrak{J}_2)$ be a linear map. Let*

$$\mathfrak{J} = \mathfrak{J}_1\oplus\mathfrak{J}_2.$$

Define the following product on $\mathfrak{J}$:

$$(x+y)(x'+y') = xx' + \pi(x)(y') + \pi(x')(y) + yy', \forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2.$$

Then $\mathfrak{J}$ is a 2SN-Jordan algebra if and only if π satisfies:

(1) $\pi(xx') = \pi(x)\pi(x') = 0$,
(2) $\pi(x)(yy') = (\pi(x)(y))y' = 0$,

for all $x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2$.

In this case, π satisfies the conditions of Definition 1.15, it is called a **2SN-admissible representation** *of $\mathfrak{J}_1$ in $\mathfrak{J}_2$ and we say that $\mathfrak{J}$ is the* **semi-direct sum** *of $\mathfrak{J}_2$ by $\mathfrak{J}_1$ by means of π.*

Proof. For all $x, x', x'' \in \mathfrak{J}_1, y, y', y'' \in \mathfrak{J}_2$, one has:

$$\begin{aligned}((x+y)(x'+y'))(x''+y'') = \pi(xx')(y'') + \pi(x'')(\pi(x)(y') + \pi(x')(y) + yy') \\ +(\pi(x)(y') + \pi(x')(y))y''.\end{aligned}$$

Therefore, $\mathfrak{J}$ is 2-step nilpotent if, and only if, $\pi(xx')$, $\pi(x)\pi(x')$, $\pi(x)(yy')$ and $(\pi(x)y)y'$ are zero, $\forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2$. □

Remark 3.8.

(1) The adjoint representation of a 2SN-Jordan algebra is an 2SN-admissible representation.
(2) Consider the particular case of $\mathfrak{J}_1 = \mathbb{C}c$ a one-dimensional algebra. If $\mathfrak{J}_1$ is 2-step nilpotent then $c^2 = 0$. Let $D := \pi(c) \in \mathrm{End}(\mathfrak{J}_2)$. The vector space $\mathfrak{J} = \mathbb{C}c\oplus\mathfrak{J}_2$ with the product:

$$(\alpha c+x)(\alpha' c+x') = \alpha D(x') + \alpha' D(x) + xx', \forall x, x' \in \mathfrak{J}_2, \alpha, \alpha' \in \mathbb{C}.$$

is a 2-step nilpotent if and only if $D^2 = 0$, $D(xx') = D(x)x' = 0, \forall x, x' \in \mathfrak{J}_2$.

(3) Let us slightly change (2) by fixing $x_0 \in \mathfrak{J}_2$ and setting the product on $\mathfrak{J} = \mathbb{C}c \oplus \mathfrak{J}_2$ as follows:

$$(\alpha c + x)(\alpha' c + x') = \alpha D(x') + \alpha' D(x) + xx' + \alpha\alpha' x_0,$$

for all $x, x' \in \mathfrak{J}_2, \alpha, \alpha' \in \mathbb{C}$. Then $\mathfrak{J}$ is a 2SN-Jordan algebra if, and only if:

$$D^2(x) = D(xx') = D(x)x' = D(x_0) = x_0 x = 0,\ \forall x, x' \in \mathfrak{J}_2.$$

In this case, we say (D, x_0) a *2SN-admissible pair* of $\mathfrak{J}_2$.

Next, we see how to obtain a 2SN-Jordan algebra from a pseudo-Euclidean one.

Proposition 3.9. *Let $(\mathfrak{J}, B)$ be a 2-step nilpotent pseudo-Euclidean Jordan algebra (or* **2SNPE-Jordan algebra** *for short), $\mathfrak{h}$ be another 2SN-Jordan algebra and $\pi : \mathfrak{h} \to \mathrm{End}_s(\mathfrak{J})$ be a linear map. Consider the bilinear map $\varphi : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{h}^*$ defined by $\varphi(x,y)(z) = B(\pi(z)(x), y), \forall x, y \in \mathfrak{J}, z \in \mathfrak{h}$. Let*

$$\overline{\mathfrak{J}} = \mathfrak{h} \oplus \mathfrak{J} \oplus \mathfrak{h}^*.$$

Define the following product on $\overline{\mathfrak{J}}$:

$$(x+y+f)(x'+y'+f') = xx' + yy' + \pi(x)(y') + \pi(x')(y) + f' \circ R_x + f \circ R_{x'} + \varphi(y,y')$$

for all $x, x' \in \mathfrak{h}, y, y' \in \mathfrak{J}, f, f' \in \mathfrak{h}^*$. Then $\overline{\mathfrak{J}}$ is a 2SN-Jordan algebra if and only if π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{J}$. Moreover, $\overline{\mathfrak{J}}$ is pseudo-Euclidean with the bilinear form

$$\overline{B}(x+y+f, x'+y'+f') = B(y,y') + f(x') + f'(x), \forall x, x' \in \mathfrak{h}, y, y' \in \mathfrak{J}, f, f' \in \mathfrak{h}^*.$$

In this case, we say that $\overline{\mathfrak{J}}$ is a **2-step nilpotent double extension** *(or* **2SN-double extension***) of $\mathfrak{J}$ by $\mathfrak{h}$ by means of π.*

Proof. If $\overline{\mathfrak{J}}$ is 2-step nilpotent then the product is commutative and $((x+y+f)(x'+y'+f'))(x''+y''+f'') = 0$ for all $x, x', x'' \in \mathfrak{h}, y, y', y'' \in \mathfrak{J}, f, f', f'' \in \mathfrak{h}^*$. By a straightforward computation, one has that π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{J}$.

Conversely, assume that π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{J}$. First, we set the extension $\mathfrak{J} \oplus \mathfrak{h}^*$ of $\mathfrak{J}$ by $\mathfrak{h}^*$ with the product:

$$(y+f)(y'+f') = yy' + \varphi(y,y'),\ \forall y, y' \in \mathfrak{J},\ f, f' \in \mathfrak{h}^*.$$

Since $\pi(z) \in \mathrm{End}_s(\mathfrak{J})$ and $\pi(z)(yy') = 0$, $\forall z \in \mathfrak{h}, y, y' \in \mathfrak{J}$, then one has φ symmetric and $\varphi(yy', y'') = 0$ for all $y, y', y'' \in \mathfrak{J}$. By Definition 3.3, $\mathfrak{J} \oplus \mathfrak{h}^*$ is a 2SN-central extension of $\mathfrak{J}$ by $\mathfrak{h}^*$.

Next, we consider the direct sum $\pi \oplus R^*$ of two representations: π and R^* of $\mathfrak{h}$ in $\mathfrak{J} \oplus \mathfrak{h}^*$ (see Definition 3.6). By a straightforward computation, we check that $\pi \oplus R^*$ satisfies the conditions of Proposition 3.7 then the semi-direct sum of $\mathfrak{J} \oplus \mathfrak{h}^*$ by $\mathfrak{h}$ by means of $\pi \oplus R^*$ is 2-step nilpotent. Finally, the product defined in $\overline{\mathfrak{J}}$ is exactly the product defined by the semi-direct sum in Proposition 3.7. Therefore we obtain the necessary and sufficient conditions.

As a consequence of Definition 1.17, $\overline{B}$ is an associative scalar product of $\overline{\mathfrak{J}}$, then $\overline{\mathfrak{J}}$ is a 2SNPE-Jordan algebra. $\square$

The notion of 2SN-double extension **does not characterize** all 2SNPE-Jordan algebras: there exist 2SNPE-Jordan algebras that can be not described in term of 2SN-double extensions, for example, the 2SNPE-Jordan algebra $\mathfrak{J} = \mathbb{C}a \oplus \mathbb{C}b$ with $a^2 = b$ and $B(a,b) = 1$, zero otherwise. Therefore, we need a better characterization given by the Proposition below, its proof is a matter of a simple calculation.

Proposition 3.10. *Let $(\mathfrak{J},B)$ be a 2SNPE-Jordan algebra, $(D,x_0) \in \mathrm{End}_s(\mathfrak{J}) \times \mathfrak{J}$ be a 2SN-admissible pair with $B(x_0,x_0) = 0$ and $(\mathfrak{t} = \mathbb{C}x_1 \oplus \mathbb{C}y_1, B_\mathfrak{t})$ be a quadratic vector space satisfying*

$$B_\mathfrak{t}(x_1,x_1) = B_\mathfrak{t}(y_1,y_1) = 0,\ B_\mathfrak{t}(x_1,y_1) = 1.$$

Fix α in $\mathbb{C}$ and consider the vector space

$$\overline{\mathfrak{J}} = \mathfrak{J} \overset{\perp}{\oplus} \mathfrak{t}$$

equipped with the product

$$y_1 \star y_1 = x_0 + \alpha x_1,\ y_1 \star x = x \star y_1 = D(x) + B(x_0,x)x_1,\ x \star y = xy + B(D(x),y)x_1$$

and $x_1 \star \overline{\mathfrak{J}} = \overline{\mathfrak{J}} \star x_1 = 0, \forall x,y \in \mathfrak{J}$. Then $\overline{\mathfrak{J}}$ is a 2SNPE-Jordan algebra with the bilinear form $\overline{B} = B + B_\mathfrak{t}$.

In this case, $(\overline{\mathfrak{J}},\overline{B})$ is called a **generalized double extension** *of $\mathfrak{J}$ by means of (D,x_0,α).*

Proposition 3.11. *Let $(\mathfrak{J},B)$ be a 2SNPE-Jordan algebra. If $\mathfrak{J}$ is non-Abelian then it is obtained from an Abelian algebra by a sequence of generalized double extensions.*

Proof. Assume that $(\mathfrak{J},B)$ is a 2SNPE-Jordan algebra and $\mathfrak{J}$ is non-Abelian. By Proposition 1.14, $\mathfrak{J}$ has a reduced ideal $\mathfrak{l}$ that is still 2-step nilpotent. That means $\mathfrak{l}^2 \neq \mathfrak{l}$, so $\mathrm{Ann}(\mathfrak{l}) \neq \{0\}$. Therefore, we can choose nonzero $x_1 \in \mathrm{Ann}(\mathfrak{l})$ such that $B(x_1,x_1) = 0$. Then there exists an isotropic element $y_1 \in \mathfrak{J}$ such that $B(x_1,y_1) = 1$. Let $\mathfrak{J} = (\mathbb{C}x_1 \oplus \mathbb{C}y_1) \overset{\perp}{\oplus} W$, where $W = (\mathbb{C}x_1 \oplus \mathbb{C}y_1)^\perp$. We have that $\mathbb{C}x_1$ and $x_1^\perp = \mathbb{C}x_1 \oplus W$ are ideals of $\mathfrak{J}$ as well.

Let $x,y \in W$, $xy = \beta(x,y) + \alpha(x,y)x_1$, where $\beta(x,y) \in W$ and $\alpha(x,y) \in \mathbb{C}$. It is easy to check that W with the product $W \times W \to W$, $(x,y) \mapsto \beta(x,y)$ is a 2SN-Jordan algebra. Moreover, it is also pseudo-Euclidean with the bilinear form $B_W = B|_{W\times W}$.

Now, we show that $\mathfrak{J}$ is a generalized double extension of (W,B_W). Indeed, let $x \in W$ then $y_1x = D(x) + \varphi(x)x_1$, where D is an endomorphism of W and $\varphi \in W^*$. Since $y_1(y_1x) = y_1(xy) = (y_1x)y = 0, \forall x,y \in W$ we get $D^2(x) = D(x)y = D(xy) = 0, \forall x,y \in W$. Moreover, $B(y_1x,y) = B(x,y_1y) = B(y_1,xy), \forall x,y \in W$ implies that $D \in \mathrm{End}_s(W)$ and $\alpha(x,y) = B_W(D(x),y), \forall x,y \in W$.

Since B_W is non-degenerate and $\varphi \in W^*$ then there exists $x_0 \in W$ such that $\varphi = B_W(x_0,.)$. Assume that $y_1y_1 = \mu y_1 + y_0 + \lambda x_1$. The equality $B(y_1y_1,x_1) = 0$ implies $\mu = 0$. Moreover, $y_0 = x_0$ since $B(y_1x,y_1) = B(x,y_1y_1), \forall x \in W$. Finally, $D(x_0) = 0$ is obtained by $y_1^3 = 0$ and this is enough to conclude that $\mathfrak{J}$ is a generalized double extension of (W,B_W) by means of (D,x_0,λ). $\square$

3.2. ***T^*-extensions of pseudo-Euclidean 2-step nilpotent.***

Given a 2SN-Jordan algebra $\mathfrak{J}$ and a symmetric bilinear map $\theta : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}^*$ such that $R^*(z)(\theta(x,y)) + \theta(xy,z) = 0,\ \forall x,y,z \in \mathfrak{J}$, then by Proposition 3.2, $\mathfrak{J} \oplus \mathfrak{J}^*$ is also a 2SN-algebra. Moreover, if θ is cyclic (that is, $\theta(x,y)(z) = \theta(y,z)(x), \forall x,y,z \in \mathfrak{J}$), then $\overline{\mathfrak{J}}$ is a pseudo-Euclidean Jordan algebra with the bilinear form defined by

$$B(x+f, y+g) = f(y) + g(x),\ \forall x,y \in \mathfrak{J},\ f,g \in \mathfrak{J}^*.$$

In a more general framework, we can define:

Definition 3.12. Let $\mathfrak{a}$ be a complex vector space and $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ a cyclic symmetric bilinear map. Assume that θ is non-degenerate, i.e. if $\theta(x,\mathfrak{a}) = 0$ then $x = 0$. Consider the vector space $\mathfrak{J} := \mathfrak{a} \oplus \mathfrak{a}^*$ equipped the product

$$(x+f)(y+g) = \theta(x,y)$$

and the bilinear form

$$B(x+f, y+g) = f(y) + g(x)$$

for all $x+f, y+g \in \mathfrak{J}$. Then $(\mathfrak{J}, B)$ is a 2SNPE-Jordan algebra and it is called the *T^*-extension* of $\mathfrak{a}$ by θ.

Lemma 3.13. *Let $\mathfrak{J}$ be a T^*-extension of $\mathfrak{a}$ by θ. If $\mathfrak{J} \neq \{0\}$ then $\mathfrak{J}$ is reduced.*

Proof. Since θ is non-degenerate, it is easy to check that $\mathrm{Ann}(\mathfrak{J}) = \mathfrak{a}^*$ is totally isotropic by the above definition. $\square$

Proposition 3.14. *Let $(\mathfrak{J}, B)$ be a 2SNPE-Jordan algebra. If $\mathfrak{J}$ is reduced then $\mathfrak{J}$ is isometrically isomorphic to some T^*-extension.*

Proof. Assume $\mathfrak{J}$ is a reduced 2SNPE-Jordan algebra. Then one has $\mathrm{Ann}(\mathfrak{J}) = \mathfrak{J}^2$, so $\dim(\mathfrak{J}^2) = \frac{1}{2}\dim(\mathfrak{J})$. Let $\mathfrak{J} = \mathrm{Ann}(\mathfrak{J}) \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a complementary subspace of $\mathrm{Ann}(\mathfrak{J})$ in $\mathfrak{J}$. Then $\mathfrak{a} \cong \mathfrak{J}/\mathfrak{J}^2$ as an Abelian algebra. Since $\mathfrak{a}$ and $\mathrm{Ann}(\mathfrak{J})$ are maximal totally isotropic subspaces of $\mathfrak{J}$, we can identify $\mathrm{Ann}(\mathfrak{J})$ to $\mathfrak{a}^*$ by the isomorphism $\varphi : \mathrm{Ann}(\mathfrak{J}) \to \mathfrak{a}^*$, $\varphi(x)(y) = B(x,y), \forall x \in \mathrm{Ann}(\mathfrak{J}), y \in \mathfrak{a}$. Define $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ by $\theta(x,y) = \varphi(xy), \forall x,y \in \mathfrak{a}$.

Now, set $\alpha : \mathfrak{J} \to \mathfrak{a} \oplus \mathfrak{a}^*$ by $\alpha(x) = p_1(x) + \varphi(p_2(x)), \forall x \in \mathfrak{J}$, where $p_1 : \mathfrak{J} \to \mathfrak{a}$ and $p_2 : \mathfrak{J} \to \mathrm{Ann}(\mathfrak{J})$ are canonical projections. Then α is isometrically isomorphic. $\square$

Proposition 3.15. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be two T^*-extensions of $\mathfrak{a}$ by θ_1 and θ_2 respectively. Then:*

(1) *there exists a Jordan algebra isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ if and only if there exist an isomorphism A_1 of $\mathfrak{a}$ and an isomorphism A_2 of $\mathfrak{a}^*$ satisfying:*

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y)), \forall x,y \in \mathfrak{a}.$$

(2) *there exists a Jordan algebra i-isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ if and only if there exists an isomorphism A_1 of $\mathfrak{a}$*

$$\theta_1(x,y) = \theta_2(A_1(x), A_1(y)) \circ A_1, \forall x,y \in \mathfrak{a}.$$

Proof.

(1) Let $A : \mathfrak{J}_1 \to \mathfrak{J}_2$ be a Jordan algebra isomorphism. Since $\mathfrak{a}^* = \mathrm{Ann}(\mathfrak{J}_1) = \mathrm{Ann}(\mathfrak{J}_2)$ is stable by A then there exist linear maps $A_1 : \mathfrak{a} \to \mathfrak{a}$, $A_1' : \mathfrak{a} \to \mathfrak{a}^*$ and $A_2 : \mathfrak{a}^* \to \mathfrak{a}^*$ such that:

$$A(x+f) = A_1(x) + A_1'(x) + A_2(f),\ \forall x+f \in \mathfrak{J}_1.$$

Since A is an isomorphism one has A_2 also isomorphic. We show that A_1 is an isomorphism of $\mathfrak{a}$. Indeed, if $A_1(x_0) = 0$ with some $x_0 \in \mathfrak{a}$ then $A(x_0) = A_1'(x_0)$ and

$$0 = A(x_0)\mathfrak{J}_2 = A(x_0 A^{-1}(\mathfrak{J}_2)) = A(x_0\mathfrak{J}_1).$$

That implies $x_0\mathfrak{J}_1 = 0$ and so $x_0 \in \mathfrak{a}^*$. That means $x_0 = 0$, i.e. A_1 is an isomorphism of $\mathfrak{a}$.

For all x and $y \in \mathfrak{a}$, one has $A(xy) = A(\theta_1(x,y)) = A_2(\theta_1(x,y))$ and

$$A(x)A(y) = (A_1(x) + A_1'(x))(A_1(y) + A_1'(y)) = A_1(x)A_1(y) = \theta_2(A_1(x),A_1(y)).$$

Therefore, $A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)), \forall x,y \in \mathfrak{a}$.

Conversely, if there exist an isomorphism A_1 of $\mathfrak{a}$ and an isomorphism A_2 of $\mathfrak{a}^*$ satisfying:

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)), \forall x,y \in \mathfrak{a},$$

then we define $A : \mathfrak{J}_1 \to \mathfrak{J}_2$ by $A(x+f) = A_1(x) + A_2(f), \forall x+f \in \mathfrak{J}_1$. It is easy to see that A is a Jordan algebra isomorphism.

(2) Assume $A : \mathfrak{J}_1 \to \mathfrak{J}_2$ is a Jordan algebra i-isomorphism then there exist A_1 and A_2 defined as in (1). Let $x \in \mathfrak{a}$, $f \in \mathfrak{a}^*$, one has:

$$B'(A(x),A(f)) = B(x,f) \Rightarrow A_2(f)(A_1(x)) = f(x).$$

Hence, $A_2(f) = f \circ A_1^{-1}, \forall f \in \mathfrak{a}^*$. Moreover, $A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y))$ implies that

$$\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)) \circ A_1, \forall x,y \in \mathfrak{a}.$$

Conversely, define $A(x+f) = A_1(x) + f \circ A_1^{-1}, \forall x+f \in \mathfrak{J}_1$ then A is an i-isomorphism.

□

Example 3.16. We keep the notations as above. Let $\mathfrak{J}'$ be the T^*-extension of $\mathfrak{a}$ by $\theta' = \lambda\theta, \lambda \neq 0$ then $\mathfrak{J}$ and $\mathfrak{J}'$ is i-isomorphic by $A : \mathfrak{J} \to \mathfrak{J}'$ defined by

$$A(x+f) = \frac{1}{\alpha}x + \alpha f, \forall x+f \in \mathfrak{J}.$$

where $\alpha \in \mathbb{C},\ \alpha^3 = \lambda$.

For a non-degenerate cyclic symmetric map θ of $\mathfrak{a}$, define a trilinear form

$$I(x,y,z) = \theta(x,y)z, \forall x,y,z \in \mathfrak{a}.$$

Then $I \in \mathsf{S}^3(\mathfrak{a})$, the space of symmetric trilinear forms on $\mathfrak{a}$. The non-degenerate condition of θ is equivalent to $\frac{\partial I}{\partial p} \neq 0, \forall p \in \mathfrak{a}^*$.

Conversely, let $\mathfrak{a}$ be a complex vector space and $I \in \mathsf{S}^3(\mathfrak{a})$ such that $\frac{\partial I}{\partial p} \neq 0$ for all $p \in \mathfrak{a}^*$. Define $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ by $\theta(x,y) := I(x,y,.), \forall x,y \in \mathfrak{a}$ then θ is symmetric and non-degenerate. Moreover, since I is symmetric, then θ is cyclic and we obtain a reduced 2SNPE-Jordan algebra $T^*_\theta(\mathfrak{a})$ defined by θ. Therefore, there is a one-to-one map from the set of all T^*-extensions of a complex vector space $\mathfrak{a}$ onto the subset $\{I \in \mathsf{S}^3(\mathfrak{a}) \mid \frac{\partial I}{\partial p} \neq 0, \forall p \in \mathfrak{a}^*\}$, such elements are also called *non-degenerate*.

Corollary 3.17. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be T^*-extensions of $\mathfrak{a}$ with respect to I_1 and I_2 non-degenerate. Then $\mathfrak{J}$ and $\mathfrak{J}'$ are i-isomorphic if and only if there exists an isomorphism A of $\mathfrak{a}$ such that*

$$I_1(x,y,z) = I_2(A(x),A(y),A(z)), \forall x,y,z \in \mathfrak{a}.$$

In particular, $\mathfrak{J}$ and $\mathfrak{J}'$ are i-isomorphic if and only if there is a isomorphism ${}^tA$ on $\mathfrak{a}^*$ which induces the isomorphism on $\mathsf{S}^3(\mathfrak{a})$, also denoted by ${}^tA$ such that ${}^tA(I_1) = I_2$. In this case, we say that I_1 and I_2 are *equivalent*.

Example 3.18. Let $\mathfrak{a} = \mathbb{C}a$ be one-dimensional vector space then $\mathsf{S}^3(\mathfrak{a}) = \mathbb{C}(a^*)^3$. By Example 3.16, T^*-extensions of $\mathfrak{a}$ by $(a^*)^3$ and $\lambda(a^*)^3,\ \lambda \neq 0$, are i-isomorphic (also, these trilinear forms are equivalent). Hence, there is only one i-isomorphic class of T^*-extensions of $\mathfrak{a}$, that is $\mathfrak{J} = \mathbb{C}a \oplus \mathbb{C}b$ with $a^2 = b$ and $B(a,b) = 1$, the other are zero.

Now, let $\mathfrak{a} = \mathbb{C}x \oplus \mathbb{C}y$ be a 2-dimensional vector space then

$$\mathsf{S}^3(\mathfrak{a}) = \{a_1(x^*)^3 + a_2(x^*)^2y^* + a_3x^*(y^*)^2 + a_4(y^*)^3, a_i \in \mathbb{C}.$$

It is easy to prove that every bivariate homogeneous polynomial of degree 3 is reducible. Therefore, by a suitable basis choice (certainly isomorphic), a non-degenerate element $I \in \mathsf{S}^3(\mathfrak{a})$ has the form $I = ax^*y^*(bx^* + cy^*),\ a,b \neq 0$. Replace $x^* := \alpha x^*$ with $\alpha^2 = ab$ to get the form $I_\lambda = x^*y^*(x^* + \lambda y^*),\ \lambda \in \mathbb{C}$.

Next, we will show that I_0 and $I_\lambda, \lambda \neq 0$ are not equivalent. Indeed, assume the contrary, i.e. there is an isomorphism ${}^tA$ such that ${}^tA(I_0) = I_\lambda$. We can write

$${}^tA(x^*) = a_1x^* + b_1y^*,\ {}^tA(y^*) = a_2x^* + b_2y^*,\ a_1,a_2,b_1,b_2 \in \mathbb{C}.$$

Then

$$\begin{aligned} {}^tA(I_0) = (a_1x^* + b_1y^*)^2(a_2x^* + b_2y^*) = a_1^2a_2(x^*)^3 + (a_1^2b_2 + 2a_1a_2b_1)(x^*)^2y^* + \\ (2a_1b_1b_2 + a_2b_1^2)x^*(y^*)^2 + b_1^2b_2(y^*)^3. \end{aligned}$$

Comparing the coefficients we will get a contradiction. Therefore, I_0 and $I_\lambda, \lambda \neq 0$ are not equivalent.

However, two forms I_{λ_1} and I_{λ_2} where $\lambda_1, \lambda_2 \neq 0$ are equivalent by the isomorphism ${}^tA$ satisfying ${}^tA(I_{\lambda_1}) = I_{\lambda_2}$ defined by:

$${}^tA(x^*) = \alpha y^*,\ {}^tA(y^*) = \beta x^*$$

where $\alpha, \beta \in \mathbb{C}$ such that $\alpha^3 = \lambda_1\lambda_2^2,\ \beta^3 = \frac{1}{\lambda_1^2\lambda_2}$. This implies that there are only two i-isomorphic classes of T^*-extensions of $\mathfrak{a}$.

Example 3.19. Let $\mathfrak{J}_0 = \mathrm{span}\{x,y,e,f\}$ be a T^*-extension of a 2-dimensional vector space $\mathfrak{a}$ by $I_0 = (x^*)^2y^*$, with $e = x^*$ and $f = y^*$, that means $B(x,e) = B(y,f) = 1$, the other are zero. It is easy to compute the product in $\mathfrak{J}_0$ defined by $x^2 = f$, $xy = e$. Let $I_\lambda = x^*y^*(x^* + \lambda y^*), \lambda \neq 0$ and $\mathfrak{J}_\lambda = \mathrm{span}\{x,y,e,f\}$ be another T^*-extension of the 2-dimensional vector space $\mathfrak{a}$ by I_λ. The products on $\mathfrak{J}_\lambda$ are $x^2 = f$, $xy = e + \lambda f$ and $yy = \lambda e$. These two algebras are neither i-isomorphic nor isomorphic. Indeed, if there is $A : \mathfrak{J}_0 \to \mathfrak{J}_\lambda$ an isomorphism. Assume $A(y) = \alpha_1 x + \alpha_2 y + \alpha_3 e + \alpha_4 f$ then

$$0 = A(yy) = (\alpha_1 x + \alpha_2 y + \alpha_3 e + \alpha_4 f)^2 = \alpha_1^2 x^2 + 2\alpha_1\alpha_2 xy + \alpha_2^2 y^2.$$

We obtain $(\lambda\alpha_2^2 + 2\alpha_1\alpha_2)e + (2\lambda\alpha_1\alpha_2 + \alpha_1^2)f = 0$. Hence, $\alpha_1 = \pm\lambda\alpha_2$. Both cases imply $\alpha_1 = \alpha_2 = 0$ (a contradiction).

We can also conclude that there are only two isomorphic classes of T^*-extensions of $\mathfrak{a}$.

4. Symmetric Novikov algebras

Definition 4.1. An algebra $\mathfrak{N}$ over $\mathbb{C}$ with a bilinear product $\mathfrak{N} \times \mathfrak{N} \to \mathfrak{N}$, $(x,y) \mapsto xy$ is called a *left-symmetric algebra* if it satisfies the identity:

$$(xy)z - x(yz) = (yx)z - y(xz), \forall x,y,z \in \mathfrak{N}. \tag{III}$$

or in terms of associators

$$(x,y,z) = (y,x,z), \forall x,y,z \in \mathfrak{N}.$$

It is called a *Novikov algebra* if in addition

$$(xy)z = (xz)y \tag{IV}$$

holds for all $x,y,z \in \mathfrak{N}$. In this case, the commutator $[x,y] := xy - yx$ of $\mathfrak{N}$ defines a Lie algebra, denoted by $\mathfrak{g}(\mathfrak{N})$, which is called the *sub-adjacent Lie algebra* of $\mathfrak{N}$. It is known that $\mathfrak{g}(\mathfrak{N})$ is a solvable Lie algebra [Bur06]. Conversely, let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[.,.]$. If there exists a bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x,y) \mapsto xy$ that satisfies (III), (IV) and $[x,y] = xy - yx, \forall x,y \in \mathfrak{J}$ then we say that $\mathfrak{g}$ admits a *Novikov structure*.

Example 4.2. Every 2-step nilpotent algebra $\mathfrak{N}$ satisfying $(xy)z = x(yz) = 0, \forall x,y,z \in \mathfrak{N}$, is a Novikov algebra.

For $x \in \mathfrak{N}$, denote by L_x and R_x respectively the left and right multiplication operators $L_x(y) = xy$, $R_x(y) = yx$, $\forall y \in \mathfrak{N}$. The condition (III) is equivalent to $[L_x, L_y] = L_{[x,y]}$ and (IV) is equivalent to $[R_x, R_y] = 0$. In the other words, the left-operators form a Lie algebra and the right-operators commute.

It is easy to check two Jacobi-type identities:

Proposition 4.3. *Let $\mathfrak{N}$ be a Novikov algebra then for all $x,y,z \in \mathfrak{N}$:*

$$[x,y]z + [y,z]x + [z,x]y = 0,$$

$$x[y,z] + y[z,x] + z[x,y] = 0.$$

Definition 4.4. Let $\mathfrak{N}$ be a Novikov algebra. A bilinear form $B : \mathfrak{N} \times \mathfrak{N} \to \mathbb{C}$ is called *associative* if

$$B(xy,z) = B(x,yz), \forall x,y,z \in \mathfrak{N}.$$

We say that $\mathfrak{N}$ is a *symmetric Novikov algebra* if it is endowed a non-degenerate associative symmetric bilinear form B.

Let $(\mathfrak{N}, B)$ be a symmetric Novikov algebra and S be a subspace of $\mathfrak{N}$. Denote by $S^\perp$ the set $\{x \in \mathfrak{N} \mid B(x,S) = 0\}$. If $B|_{S\times S}$ is non-degenerate (resp. degenerate) then we say that S is *non-degenerate* (resp. *degenerate*).

The proof of Lemma 4.5 and Proposition 4.6 below is lengthy, but straight forward then we omit it.

Lemma 4.5. *Let $(\mathfrak{N}, B)$ be a symmetric Novikov algebra and I be an ideal of $\mathfrak{N}$ then*

(1) *$I^\perp$ is also an ideal of $\mathfrak{N}$ and $II^\perp = I^\perp I = \{0\}$*
(2) *If I is non-degenerate then so is $I^\perp$ and $\mathfrak{N} = I \overset{\perp}{\oplus} I^\perp$.*

Proposition 4.6. *We call the set $C(\mathfrak{N}) := \{x \in \mathfrak{N} \mid xy = yx, \forall y \in \mathfrak{N}\}$ the* **center** *of $\mathfrak{N}$ and denote by $As(\mathfrak{N}) = \{x \in \mathfrak{N} \mid (x,y,z) = 0, \forall y,z \in \mathfrak{N}\}$. One has*

(1) *If $\mathfrak{N}$ is a Novikov algebra then $C(\mathfrak{N}) \subset N(\mathfrak{N})$, where $N(\mathfrak{N})$ is the nucleus of $\mathfrak{N}$ defined in Definition 1.11 (3). Moreover, if $\mathfrak{N}$ is also commutative then $N(\mathfrak{N}) = \mathfrak{N} = As(\mathfrak{N})$ (that means $\mathfrak{N}$ is an associative algebra).*
(2) *If $(\mathfrak{N}, B)$ is a symmetric Novikov algebra then*
   (i) $C(\mathfrak{N}) = [\mathfrak{g}(\mathfrak{N}), \mathfrak{g}(\mathfrak{N})]^\perp$.
   (ii) $N(\mathfrak{N}) = As(\mathfrak{N}) = (\mathfrak{N}, \mathfrak{N}, \mathfrak{N})^\perp$.
   (iii) $L\,\mathrm{Ann}(\mathfrak{N}) = R\,\mathrm{Ann}(\mathfrak{N}) = \mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}\mathfrak{N})^\perp$.

Proposition 4.7. *Let $\mathfrak{N}$ be a Novikov algebra then*

(1) *$C(\mathfrak{N})$ is a commutative subalgebra.*
(2) *$As(\mathfrak{N})$, $N(\mathfrak{N})$ are ideals.*

Proof.

(1) Let $x,y \in C(\mathfrak{N})$ then $(xy)z = (xz)y = (zx)y = z(xy) + (z,x,y) = z(xy), \forall z \in \mathfrak{N}$. Therefore, $xy \in C(\mathfrak{N})$ and then $C(\mathfrak{N})$ is a subalgebra of $\mathfrak{N}$. Certainly, $C(\mathfrak{N})$ is commutative.
(2) Let $x \in As(\mathfrak{N}), y,z,t \in \mathfrak{N}$. By the equality

$$(xy,z,t) = ((xy)z)t - (xy)(zt) = ((xz)t)y - (x(zt))y = (x,z,t)y = 0,$$

one has $xy \in As(\mathfrak{N})$. Moreover,

$$\begin{aligned}(yx,z,t) &= ((yx)z)t - (yx)(zt) = (y(xz))t - y(x(zt)) \\ &= (y,xz,t) + y((xz)t) - y(x(zt)) = y(x,z,t) = 0\end{aligned}$$

since $xz \in As(\mathfrak{N})$. Therefore $As(\mathfrak{N})$ is an ideal of $\mathfrak{N}$.

Similarly, let $x \in N(\mathfrak{N}), y,z,t \in \mathfrak{N}$ one has:

$$\begin{aligned}(y,z,xt) &= (yz)(xt) - y(z(xt)) = ((yz)x)t - (yz,x,t) - y((zx)t - (z,x,t)) \\ &= ((yz)x)t - (y(zx))t + (y,zx,t) = (y,z,x)t = 0\end{aligned}$$

and

$$(y,z,tx) = (yz)(tx) - y(z(tx)) = ((yz)t)x - (yz,t,x) - y((zt)x - (z,t,x))$$
$$= ((yz)x)t - y((zx)t) = (y,z,x)t + (y,zx,t) = 0.$$

Then $N(\mathfrak{N})$ is also an ideal of $\mathfrak{N}$.

□

Lemma 4.8. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then $[L_x,L_y] = L_{[x,y]} = 0$ for all $x,y \in \mathfrak{N}$. Consequently, for a symmetric Novikov algebra, the Lie algebra formed by the left-operators is Abelian.*

Proof. It follows the proof of Lemma II.5 in [AB10]. Fix $x,y \in \mathfrak{N}$, for all $z,t \in \mathfrak{N}$ one has

$$B([L_x,L_y](z),t) = B(x(yz) - y(xz),t) = B((tx)y - (ty)x,z) = 0.$$

Therefore, $[L_x,L_y] = L_{[x,y]} = 0, \forall x,y \in \mathfrak{N}$. □

Corollary 4.9. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then the sub-adjacent Lie algebra $\mathfrak{g}(\mathfrak{N})$ of $\mathfrak{N}$ with the bilinear form B becomes a quadratic 2-step nilpotent Lie algebra.*

Proof. One has

$$B([x,y],z) = B(xy - yx,z) = B(x,yz) - B(x,zy) = B(x,[y,z]), \forall x,y,z \in \mathfrak{N}.$$

Hence, $\mathfrak{g}(\mathfrak{N})$ is quadratic. By Lemma 4.8 and 2(iii) of Proposition 4.6 one has $[x,y] \in L\,\mathrm{Ann}(\mathfrak{N}) = \mathrm{Ann}(\mathfrak{N})$, $\forall x,y \in \mathfrak{N}$. That implies $[[x,y],z] \in \mathrm{Ann}(\mathfrak{N})\mathfrak{N} = \{0\}, \forall x,y \in \mathfrak{N}$, i.e. $\mathfrak{g}(\mathfrak{N})$ is 2-step nilpotent. □

It results that the classification of quadratic 2-step nilpotent Lie algebras ([Ova07], [Duo10]) is closely related to the classification of symmetric Novikov algebras. For instance, by [DPU], every quadratic 2-step nilpotent Lie algebra of dimension ≤ 5 is Abelian so that every symmetric Novikov algebra of dimension ≤ 5 is commutative. In general, in the case of dimension ≥ 6, there exists a non-commutative symmetric Novikov algebra by Proposition 4.11 below.

Definition 4.10. Let $\mathfrak{N}$ be a Novikov algebra. We say that $\mathfrak{N}$ is an *anti-commutative Novikov algebra* if

$$xy = -yx, \forall x,y \in \mathfrak{N}.$$

Proposition 4.11. *Let $\mathfrak{N}$ be a Novikov algebra. Then $\mathfrak{N}$ is anti-commutative if, and only if, $\mathfrak{N}$ is a 2-step nilpotent Lie algebra with the Lie bracket defined by $[x,y] := xy, \forall x,y \in \mathfrak{N}$.*

Proof. Assume that $\mathfrak{N}$ is a Novikov algebra such that $xy = -yx, \forall x,y \in \mathfrak{N}$. Since the commutator $[x,y] = xy - yx = 2xy$ is a Lie bracket, so the product $(x,y) \mapsto xy$ is also a Lie bracket. The identity (III) of Definition 4.1 is equivalent to $(xy)z = 0, \forall x,y,z \in \mathfrak{N}$. It shows that $\mathfrak{N}$ is a 2-step nilpotent Lie algebra.

Conversely, if $\mathfrak{N}$ is a 2-step nilpotent Lie algebra then we define the product $xy := [x,y], \forall x,y \in \mathfrak{N}$. It is obvious that the identities (III) and (IV) of Definition 4.1 are satisfied since $(xy)z = 0, \forall x,y,z \in \mathfrak{N}$. □

By the above Proposition, the study of anti-commutative Novikov algebras is reduced to the study of 2-step nilpotent Lie algebras. Moreover, the formula in this proposition also can be used to define a 2-step nilpotent symmetric Novikov algebra from a quadratic 2-step nilpotent Lie algebra. Recall that there exists only one non-Abelian quadratic 2-step nilpotent Lie algebra of dimension 6 up to isomorphism [DPU] then there is only one anti-commutative symmetric Novikov algebra of dimension 6 up to isomorphism. However, there exist non-commutative symmetric Novikov algebras that are not 2-step nilpotent [AB10]. For instance, let $\mathfrak{N} = \mathfrak{g}_6 \overset{\perp}{\oplus} \mathbb{C}c$, where $\mathfrak{g}_6$ is the 6-dimensional elementary quadratic Lie algebra [DPU] and $\mathbb{C}c$ is a pseudo-Euclidean simple Jordan algebra with the bilinear form $B_c(c,c) = 1$ (obviously, this algebra is a symmetric Novikov algebra and commutative). Then $\mathfrak{N}$ become a symmetric Novikov algebra with the bilinear form defined by $B = B_{\mathfrak{g}_6} + B_c$, where $B_{\mathfrak{g}_6}$ is the bilinear form on $\mathfrak{g}_6$. We can extend this example for the case $\mathfrak{N} = \mathfrak{g} \overset{\perp}{\oplus} \mathfrak{J}$, where $\mathfrak{g}$ is a quadratic 2-step nilpotent Lie algebra and $\mathfrak{J}$ is a symmetric Jordan-Novikov algebra defined below. However, these algebras are decomposable. An example in the indecomposable case of dimension 7 can be found in the last part of this paper.

Proposition 4.12. *Let $\mathfrak{N}$ be a Novikov algebra. Assume that its product is commutative, that means $xy = yx, \forall x,y \in \mathfrak{N}$. Then the identities (III) and (IV) of Definition 4.1 are equivalent to the only condition:*

$$(x,y,z) = (xy)z - x(yz) = 0, \forall x,y,z \in \mathfrak{N}.$$

It means that $\mathfrak{N}$ is an associative algebra. Moreover, $\mathfrak{N}$ is also a Jordan algebra. In this case, we say that $\mathfrak{N}$ is a **Jordan-Novikov algebra**. *In addition, if $\mathfrak{N}$ has a non-degenerate associative symmetric bilinear form, then we say that $\mathfrak{N}$ is a* symmetric *Jordan-Novikov algebra.*

Proof. Assume $\mathfrak{N}$ is a commutative Novikov algebra. By (1) of Proposition 4.6, the product is also associative. Conversely, if one has the condition:

$$(xy)z - x(yz) = 0, \forall x,y,z \in \mathfrak{N}$$

then (III) identifies with zero and (IV) is obtained by $(yx)z = y(xz), \forall x,y,z \in \mathfrak{N}$. □

Example 4.13. Recall the pseudo-Euclidean Jordan algebra $\mathfrak{J}$ in Example 2.10 spanned by $\{x, x_1, y_1\}$, where the commutative product on $\mathfrak{J}$ is defined by:

$$y_1^2 = y_1, y_1 x = x, y_1 x_1 = x_1, x^2 = x_1.$$

It is easy to check that this product is also associative. Therefore, $\mathfrak{J}$ is a symmetric Jordan-Novikov algebra with the bilinear form B defined $B(x_1,y_1) = B(x,x) = 1$ and the other zero.

Example 4.14. Pseudo-Euclidean 2-step nilpotent Jordan algebras are symmetric Jordan-Novikov algebras.

Remark 4.15.

(1) By Lemma 4.8, if the symmetric Novikov algebra $\mathfrak{N}$ has $\mathrm{Ann}(\mathfrak{N}) = \{0\}$ then $[x,y] = xy - yx = 0, \forall x,y \in \mathfrak{N}$. It implies that $\mathfrak{N}$ is commutative and then $\mathfrak{N}$ is a symmetric Jordan-Novikov algebra.
(2) If the product on $\mathfrak{N}$ is associative then it may not be commutative. An example can be found in the next part.
(3) Let $\mathfrak{N}$ be a Novikov algebra with unit element e; that is $ex = xe = x, \forall x \in \mathfrak{N}$. Then $xy = (ex)y = (ey)x = yx,\ \forall x,y \in \mathfrak{N}$ and therefore $\mathfrak{N}$ is a Jordan-Novikov algebra.
(4) The algebra given in Example 4.13 is also a Frobenius algebra, that is, a finite-dimensional associative algebra with unit element equipped with a non-degenerate associative bilinear form.

A well-known result is that every associative algebra $\mathfrak{N}$ is Lie-admissible and Jordan-admissible; that is, if $(x,y) \mapsto xy$ is the product of $\mathfrak{N}$ then the products

$$[x,y] = xy - yx \qquad \text{and}$$

$$[x,y]_+ := xy + yx$$

define respectively a Lie algebra structure and a Jordan algebra structure on $\mathfrak{N}$. There exist algebras satisfying each one of these properties. For example, the non-commutative Jordan algebras are Jordan-admissible [Sch55] or the Novikov algebras are Lie-admissible. However, remark that a Novikov algebra may not be Jordan-admissible by the following example:

Example 4.16. Consider the 2-dimensional algebra $\mathfrak{N} = \mathbb{C}a \oplus \mathbb{C}b$ such that $ba = -a$, zero otherwise. Then $\mathfrak{N}$ is a Novikov algebra [BMH02]. One has $[a,b] = a$ and $[a,b]_+ = -a$. For $x \in \mathfrak{N}$, denote by ad^+_x the endomorphism of $\mathfrak{N}$ defined by $\mathrm{ad}^+_x(y) = [x,y]_+ = [y,x]_+,\ \forall y \in \mathfrak{N}$. It is easy to see that

$$\mathrm{ad}^+_a = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}^+_b = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $x = \lambda a + \mu b \in \mathfrak{N},\ \lambda, \mu \in \mathbb{C}$, one has $[x,x]_+ = -2\lambda\mu a$ and therefore:

$$\mathrm{ad}^+_x = \begin{pmatrix} -\mu & -\lambda \\ 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}^+_{[x,x]_+} = \begin{pmatrix} 0 & 2\lambda\mu \\ 0 & 0 \end{pmatrix}.$$

Since $[\mathrm{ad}^+_x, \mathrm{ad}^+_{[x,x]_+}] \neq 0$ if $\lambda, \mu \neq 0$, then $\mathfrak{N}$ is not Jordan-admissible.

We will give a condition for a Novikov algebra to be Jordan-admissible as follows:

Proposition 4.17. *Let $\mathfrak{N}$ be a Novikov algebra satisfying*

$$\text{(V)} \qquad (x,x,x) = 0, \forall x \in \mathfrak{N}.$$

Define on $\mathfrak{N}$ the product $[x,y]_+ = xy + yx, \forall x,y \in \mathfrak{N}$ then $\mathfrak{N}$ is a Jordan algebra with this product. In this case, it is called the **associated Jordan algebra** *of $\mathfrak{N}$ and denoted by $\mathfrak{J}(\mathfrak{N})$.*

Proof. Let $x, y \in \mathfrak{N}$ then we can write $x^3 = x^2x = xx^2$. One has

$$\begin{aligned}[][x,y]_+,[x,x]_+]_+ &= [xy+yx, 2x^2]_+\\ &= 2(xy)x^2 + 2(yx)x^2 + 2x^2(xy) + 2x^2(yx)\\ &= 2x^3y + 2(yx)x^2 + 2x^2(xy) + 2x^2(yx)\end{aligned}$$

and

$$\begin{aligned}[][x,[y,[x,x]_+]_+]_+ &= [x, 2yx^2 + 2x^2y]_+\\ &= 2x(yx^2) + 2x(x^2y) + 2(yx^2)x + 2(x^2y)x\\ &= 2x(yx^2) + 2x(x^2y) + 2(yx)x^2 + 2x^3y.\end{aligned}$$

Therefore, $[[x,y]_+,[x,x]_+]_+ = [x,[y,[x,x]_+]_+]_+$ if and only if $x^2(xy) + x^2(yx) = x(yx^2) + x(x^2y)$. Remark that we have following identities:

$$\begin{aligned}x^2(xy) &= x^3y - (x^2,x,y) = x^3y - (x,x^2,y),\\ x^2(yx) &= (x^2y)x - (x^2,y,x) = x^3y - (y,x^2,x),\\ x(yx^2) &= (xy)x^2 - (x,y,x^2) = x^3y - (y,x,x^2),\\ x(x^2y) &= x^3y - (x,x^2,y).\end{aligned}$$

It means that we have only to check the formula $(y,x^2,x) = (y,x,x^2)$. It is clear by the identities (III) and (V). Then we can conclude that $\mathfrak{J}(\mathfrak{N})$ is a Jordan algebra. $\square$

Corollary 4.18. *If $(\mathfrak{N},B)$ is a symmetric Novikov algebra satisfying (V) then $(\mathfrak{J}(\mathfrak{N}),B)$ is a pseudo-Euclidean Jordan algebra.*

Proof. It is obvious since $B([x,y]_+,z) = B(xy+yx,z) = B(x,yz+zy) = B(x,[y,z]_+)$, $\forall x,y,z \in \mathfrak{J}(\mathfrak{N})$. $\square$

Remark 4.19. Obviously, Jordan-Novikov algebras are power-associative but in general this is not true for Novikov algebras. Indeed, if Novikov algebras were power-associative then they would satisfy (V). That would imply they were Jordan-admissible. But, that is a contradiction as shown in Example 4.16.

Lemma 4.20. *Let $\mathfrak{N}$ be a Novikov algebra then $[x,yz]_+ = [y,xz]_+$, $\forall x,y,z \in \mathfrak{N}$.*

Proof. By (III), for all $x,y,z \in \mathfrak{N}$ one has $(xy)z + y(xz) = x(yz) + (yx)z$. Combine with (IV), we obtain:

$$(xz)y + y(xz) = x(yz) + (yz)x.$$

That means $[x,yz]_+ = [y,xz]_+$, $\forall x,y,z \in \mathfrak{N}$. $\square$

Proposition 4.21. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then following identities:*

(1) $x[y,z] = [y,z]x = 0$. *Consequently,* $[x,yz]_+ = [x,zy]_+$.
(2) $[x,y]_+z = [x,z]_+y$,
(3) $[x,yz]_+ = [xy,z]_+ = x[y,z]_+ = [x,y]_+z$,
(4) $x[y,z]_+ = [y,z]_+x$.

hold for all $x,y,z \in \mathfrak{N}$.

Proof. Let x,y,z,t be elements $\in \mathfrak{N}$,

(1) By Proposition 4.6 and Lemma 4.8, $L_{[y,z]} = 0$ so one has (1).
(2) $B([x,y]_+z,t) = B(y,[x,zt]_+) = B(y,[z,xt]_+) = B([z,y]_+x,t)$. Therefore, $[x,y]_+z = [z,y]_+x$. Since the product $[.,.]_+$ is commutative then $[y,x]_+z = [y,z]_+x$.
(3) By (1) and Lemma 4.20, $[x,yz]_+ = [x,zy]_+ = [z,xy]_+ = [xy,z]_+$.
Since B is associative with respect to the product in $\mathfrak{N}$ and in $\mathfrak{J}(\mathfrak{N})$ then

$$B(t,[xy,z]_+) = B([t,xy]_+,z) = B([t,yx]_+,z) = B([y,tx]_+,z) = B(tx,[y,z]_+) = B(t,x[y,z]_+).$$

It implies that $[xy,z]_+ = x[y,z]_+$. Similarly,

$$B([x,y]_+z,t) = B(x,[y,zt]_+) = B(x,[y,tz]_+) = B(x,[t,yz]_+) = B([x,yz]_+,t).$$

So $[x,y]_+z = [x,yz]_+$.
(4) By (2) and (3), $x[y,z]_+ = [x,y]_+z = [y,x]_+z = [y,z]_+x$.

$\square$

Corollary 4.22. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then $(\mathfrak{J}(\mathfrak{N}),B)$ is a symmetric Jordan-Novikov algebra.*

Proof. We will show that $[[x,y]_+,z]_+ = [x,[y,z]_+]_+$, $\forall x,y,z \in \mathfrak{N}$. Indeed, By Proposition 4.21 one has

$$[[x,y]_+,z]_+ = [2xy,z]_+ = 2[z,xy]_+ = 2[x,yz]_+ = [x,[y,z]_+]_+.$$

Hence, the product $[.,.]_+$ are both commutative and associative. That means $\mathfrak{J}(\mathfrak{N})$ be a Jordan-Novikov algebra. $\square$

It results that for symmetric Novikov algebras the condition (V) is not necessary. Moreover, we have the much stronger fact as follows:

Proposition 4.23. *Let $\mathfrak{N}$ be a symmetric Novikov algebra then the product on $\mathfrak{N}$ is associative, that is $x(yz) = (xy)z, \forall x,y,z \in \mathfrak{N}$.*

Proof. Firstly, we need the lemma:

Lemma 4.24. *Let $\mathfrak{N}$ be a symmetric Novikov algebra then $\mathfrak{N}\mathfrak{N} \subset C(\mathfrak{N})$.*

Proof. By Lemma 4.8, one has $[x,y] = xy - yx \in \mathrm{Ann}(\mathfrak{N}) \subset C(\mathfrak{N}), \forall x,y \in \mathfrak{N}$. Also, by (4) of Proposition 4.21, $x[y,z]_+ = [y,z]_+x$, $\forall x,y,z \in \mathfrak{N}$, that means $[x,y]_+ = xy + yx \in C(\mathfrak{N}), \forall x,y \in C(\mathfrak{N})$. Hence, $xy \in C(\mathfrak{N}), \forall x,y \in C(\mathfrak{N})$, i.e. $\mathfrak{N}\mathfrak{N} \subset C(\mathfrak{N})$. $\square$

Let $x,y,z \in \mathfrak{N}$. By above Lemma, one has $(yz)x = x(yz)$. Combine with (IV), $(yx)z = x(yz)$. On the other hand, $[x,y] \in \mathrm{Ann}(\mathfrak{N})$ implies $(yx)z = (xy)z$. Therefore, $(xy)z = x(yz)$. $\square$

A general proof of the above Proposition can be found in [AB10], Lemma II.4 which holds for all symmetric left-symmetric superalgebras.

By Corollary 4.9, if $\mathfrak{N}$ is a symmetric Novikov algebra then $\mathfrak{g}(\mathfrak{N})$ is 2-step nilpotent. However, $\mathfrak{J}(\mathfrak{N})$ is not necessarily 2-step nilpotent, for example the one-dimensional Novikov algebra $\mathbb{C}c$ with $c^2 = c$ and $B(c,c) = 1$. If $\mathfrak{N}$ is a symmetric 2-step nilpotent Novikov algebra then $(xy)z = 0, \forall x,y,z \in \mathfrak{N}$. So $[[x,y]_+,z]_+ = 0$, $\forall x,y,z \in \mathfrak{N}$. That implies $\mathfrak{J}(\mathfrak{N})$ is also a 2-step nilpotent Jordan algebra. The converse is also true.

Proposition 4.25. *Let $\mathfrak{N}$ be a symmetric Novikov algebra. If $\mathfrak{J}(\mathfrak{N})$ is a 2-step nilpotent Jordan algebra then $\mathfrak{N}$ is a 2-step nilpotent Novikov algebra.*

Proof. Since (4) of Proposition 4.21, if $x,y,z \in \mathfrak{N}$ then one has

$$[[x,y]_+,z]_+ = [x,y]_+z + z[x,y]_+ = 2[x,y]_+z = 0.$$

It means $[x,y]_+ = xy + yx \in \mathrm{Ann}(\mathfrak{N})$. On the other hand, $[x,y] = xy - yx \in \mathrm{Ann}(\mathfrak{N})$ then $xy \in \mathrm{Ann}(\mathfrak{N}), \forall x,y \in \mathfrak{N}$. Therefore, $\mathfrak{N}$ is 2-step nilpotent. □

By Proposition 4.11, since the lowest dimension of non-Abelian quadratic 2-step nilpotent Lie algebras is six then examples of symmetric non-commutative Novikov algebras must be at least six dimensional. One of those can be found in [ZC07] and it is also described in term of double extension in [AB10]. We recall this algebra as follows:

Example 4.26. Firstly, we define the **character matrix** of a Novikov algebra $\mathfrak{N} = \mathrm{span}\{e_1,\dots,e_n\}$ by

$$\begin{pmatrix} \sum_k c_{11}^k e_k & \dots & \sum_k c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum_k c_{n1}^k e_k & \dots & \sum_k c_{nn}^k e_k \end{pmatrix},$$

where c_{ij}^k are the **structure constants** of $\mathfrak{N}$, i. e. $e_ie_j = \sum_k c_{ij}^k e_k$.

Now, let $\mathfrak{N}_6$ be a 6-dimensional vector space spanned by $\{e_1,\dots,e_6\}$ then $\mathfrak{N}_6$ is a symmetric non-commutative Novikov algebras with character matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_2 & 0 & 0 \end{pmatrix}$$

and the bilinear form B defined by:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Obviously, in this case, $\mathfrak{N}_6$ is a 2-step nilpotent Novikov algebra with $\mathrm{Ann}(\mathfrak{N}) = \mathfrak{N}\mathfrak{N}$. Moreover, $\mathfrak{N}_6$ is indecomposable since it is non-commutative and all of symmetric Novikov algebras up to dimension 5 are commutative.

We need the following lemma:

Lemma 4.27. *Let $\mathfrak{N}$ be a non-Abelian symmetric Novikov algebra then $\mathfrak{N} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$, where $\mathfrak{z} \subset \mathrm{Ann}(\mathfrak{N})$ and $\mathfrak{l}$ is a reduced symmetric Novikov algebra, that means $\mathfrak{l} \neq \{0\}$ and $\mathrm{Ann}(\mathfrak{l}) \subset \mathfrak{l}\mathfrak{l}$.*

Proof. Let $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{N}) \cap \mathfrak{N}\mathfrak{N}$, $\mathfrak{z}$ is a complementary subspace of $\mathfrak{z}_0$ in $\mathrm{Ann}(\mathfrak{N})$ and $\mathfrak{l} = (\mathfrak{z})^{\perp}$. If x is an element in $\mathfrak{z}$ such that $B(x,\mathfrak{z}) = 0$ then $B(x,\mathfrak{N}\mathfrak{N}) = 0$ since $\mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}\mathfrak{N})^{\perp}$. As a consequence, $B(x,\mathfrak{z}_0) = 0$ and then $B(x,\mathrm{Ann}(\mathfrak{N})) = 0$. Hence, x must be in $\mathfrak{N}\mathfrak{N}$ since $\mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}\mathfrak{N})^{\perp}$. It shows that $x = 0$ and $\mathfrak{z}$ is non-degenerate. By Lemma 4.5, $\mathfrak{l}$ is a non-degenerate ideal and $\mathfrak{N} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$.

Since $\mathfrak{N}$ is non-Abelian then $\mathfrak{l} \neq \{0\}$. Moreover, $\mathfrak{l}\mathfrak{l} = \mathfrak{N}\mathfrak{N}$ implies $\mathfrak{z}_0 \subset \mathfrak{l}\mathfrak{l}$. It is easy to see that $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{l})$ and the lemma is proved. □

Proposition 4.28. *Let $\mathfrak{N}$ be a symmetric non-commutative Novikov algebras of dimension 6 then $\mathfrak{N}$ is 2-step nilpotent.*

Proof. Let $\mathfrak{N} = \mathrm{span}\{x_1,x_2,x_3,z_1,z_2,z_3\}$. By [DPU], there exists only one non-Abelian quadratic 2-step nilpotent Lie algebra of dimension 6 (up to isomorphisms) then $\mathfrak{g}(\mathfrak{N}) = \mathfrak{g}_6$. We can choose the basis such that $[x_1,x_2] = z_3$, $[x_2,x_3] = z_1$, $[x_3,x_1] = z_2$ and the bilinear form $B(x_i,z_i) = 1,\ i = 1,2,3$, the other are zero.

Recall that $C(\mathfrak{N}) := \{x \in \mathfrak{N} \mid xy = yx, \forall y \in \mathfrak{N}\}$ then $C(\mathfrak{N}) = \{x \in \mathfrak{N} \mid [x,y] = 0, \forall y \in \mathfrak{N}\}$. Therefore, $C(\mathfrak{N}) = \mathrm{span}\{z_1,z_2,z_3\}$ and $\mathfrak{N}\mathfrak{N} \subset C(\mathfrak{N})$ by Lemma 4.24. Consequently, $\dim(\mathfrak{N}\mathfrak{N}) \leq 3$.

By the above lemma, if $\mathfrak{N}$ is not reduced then $\mathfrak{N} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ with $\mathfrak{z} \subset \mathrm{Ann}(\mathfrak{N})$ is a non-degenerate ideal and $\mathfrak{z} \neq \{0\}$. It implies that $\mathfrak{l}$ is a symmetric Novikov algebra having dimension ≤ 5 and then $\mathfrak{l}$ is commutative. This is a contradiction since $\mathfrak{N}$ is non-commutative. Therefore, $\mathfrak{N}$ must be reduced and $\mathrm{Ann}(\mathfrak{N}) \subset \mathfrak{N}\mathfrak{N}$. Moreover, $\dim(\mathfrak{N}\mathfrak{N}) + \dim(\mathrm{Ann}(\mathfrak{N})) = 6$ so we have $\mathfrak{N}\mathfrak{N} = \mathrm{Ann}(\mathfrak{N}) = C(\mathfrak{N})$. It shows $\mathfrak{N}$ is 2-step nilpotent. □

In this case, the character matrix of $\mathfrak{N}$ in the basis $\{x_1,x_2,x_3,z_1,z_2,z_3\}$ is given by:

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is a 3×3-matrix defined by the structure constants $x_i x_j = \sum_k c_{ij}^k z_k$, $1 \leq i,j,k \leq 3$, and B has the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $B(x_i x_j, x_r) = B(x_i, x_j x_r) = B(x_j, x_r x_i)$ then one has $c_{ij}^r = c_{jr}^i = c_{ri}^j$, $1 \leq i,j,k \leq 3$.

Next, we give some simple properties for symmetric Novikov algebras as follows:

Proposition 4.29. *Let $\mathfrak{N}$ be a symmetric non-commutative Novikov algebra. If $\mathfrak{N}$ is reduced then*

$$3 \leq \dim(\mathrm{Ann}(\mathfrak{N})) \leq \dim(\mathfrak{N}\mathfrak{N}) \leq \dim(\mathfrak{N}) - 3.$$

Proof. By Lemma 4.24, $\mathfrak{N}\mathfrak{N} \subset C(\mathfrak{N})$. Moreover, $\mathfrak{N}$ non-commutative implies that $\mathfrak{g}(\mathfrak{N})$ is non-Abelian and by [PU07], $\dim([\mathfrak{N},\mathfrak{N}]) \geq 3$. Therefore, $\dim C(\mathfrak{N}) \leq \dim(\mathfrak{N}) - 3$ since $C(\mathfrak{N}) = [\mathfrak{N},\mathfrak{N}]^{\perp}$. Consequently, $\dim(\mathfrak{N}\mathfrak{N}) \leq \dim(\mathfrak{N}) - 3$ and then $\dim(\mathrm{Ann}(\mathfrak{N})) \geq 3$. □

Corollary 4.30. *Let $\mathfrak{N}$ be a symmetric non-commutative Novikov algebra of dimension 7. If $\mathfrak{N}$ is 2-step nilpotent then $\mathfrak{N}$ is not reduced.*

Proof. Assume that $\mathfrak{N}$ is reduced then $\dim(\mathrm{Ann}(\mathfrak{N})) = 3$ and $\dim(\mathfrak{N}\mathfrak{N}) = 4$. It implies that there must have a nonzero element $x \in \mathfrak{N}\mathfrak{N}$ such that $x\mathfrak{N} \neq \{0\}$ and then $\mathfrak{N}$ is not 2-step nilpotent. □

Now, we give a more general result for symmetric Novikov algebra of dimension 7 as follows:

Proposition 4.31. *Let $\mathfrak{N}$ be a symmetric non-commutative Novikov algebra of dimension 7. If $\mathfrak{N}$ is reduced then there are only two cases:*

(1) *$\mathfrak{N}$ is 3-step nilpotent and indecomposable.*
(2) *$\mathfrak{N}$ is decomposable by $\mathfrak{N} = \mathbb{C}x \overset{\perp}{\oplus} \mathfrak{N}_6$, where $x^2 = x$ and $\mathfrak{N}_6$ is a symmetric non-commutative Novikov algebra of dimension 6.*

Proof. Assume that $\mathfrak{N}$ is reduced then $\dim(\mathrm{Ann}(\mathfrak{N})) = 3$, $\dim(\mathfrak{N}\mathfrak{N}) = 4$ since $\mathrm{Ann}(\mathfrak{N}) \subset \mathfrak{N}\mathfrak{N}$ and $\mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}\mathfrak{N})^{\perp}$. By [Bou59], $\mathrm{Ann}(\mathfrak{N})$ is totally isotropic, then there exist a totally isotropic subspace V and a nonzero x of $\mathfrak{N}$ such that

$$\mathfrak{N} = \mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x \oplus V,$$

where $\mathrm{Ann}(\mathfrak{N}) \oplus V$ is non-degenerate, $B(x,x) \neq 0$ and $x^{\perp} = \mathrm{Ann}(\mathfrak{N}) \oplus V$. As a consequence, $\mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x = (\mathrm{Ann}(\mathfrak{N}))^{\perp} = \mathfrak{N}\mathfrak{N}$.

Consider the left-multiplication operator $L_x : \mathbb{C}x \oplus V \to \mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x$, $L_x(y) = xy$, $\forall y \in \mathbb{C}x \oplus V$. Denote by p the projection $\mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x \to \mathbb{C}x$.

- If $p \circ L_x = 0$ then $(\mathfrak{N}\mathfrak{N})\mathfrak{N} = x\mathfrak{N} \subset \mathrm{Ann}(\mathfrak{N})$. Therefore, $((\mathfrak{N}\mathfrak{N})\mathfrak{N})\mathfrak{N} = \{0\}$. That implies $\mathfrak{N}$ is 3-nilpotent. If $\mathfrak{N}$ is decomposable then $\mathfrak{N}$ must be 2-step nilpotent. This is in contradiction to Corollary 4.30.
- If $p \circ L_x \neq 0$ then there is a nonzero $y \in \mathbb{C}x \oplus V$ such that $xy = ax + z$ with $0 \neq a \in \mathbb{C}$ and $z \in \mathrm{Ann}(\mathfrak{N})$. In this case, we can choose y such that $a = 1$. It implies that $(x^2)y = x(xy) = x^2$.

  If $x^2 = 0$ then $0 = B(x^2,y) = B(x,xy) = B(x,x)$. This is a contradiction. Therefore, $x^2 \neq 0$. Since $x^2 \in \mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x$ then $x^2 = z' + \mu x$, where $z' \in \mathrm{Ann}(\mathfrak{N})$ and $\mu \in \mathbb{C}$ must be nonzero. By setting $x' := \frac{x}{\mu}$ and $z'' = \frac{z'}{\mu^2}$, we get $(x')^2 = z'' + x'$. Let $x_1 := (x')^2$, one has:

  $$x_1^2 = (x')^2(x')^2 = (z'' + x')(z'' + x') = x_1.$$

Moreover, for all $t = \lambda x + v \in \mathbb{C}x \oplus V$, we have $t(x^2) = (x^2)t = x(xt) = \lambda\mu(x^2)$. It implies that $\mathbb{C}x^2 = \mathbb{C}x_1$ is an ideal of $\mathfrak{N}$.

Since $B(x_1, x_1) \neq 0$, by Lemma 4.5 one has $\mathfrak{N} = \mathbb{C}x_1 \overset{\perp}{\oplus} (x_1)^\perp$. Certainly, $(x_1)^\perp$ is a symmetric non-commutative Novikov algebra of dimension 6.

□

Proposition 4.32. *Let $\mathfrak{N}$ be a symmetric Novikov algebra. If $\mathfrak{g}(\mathfrak{N})$ or $\mathfrak{J}(\mathfrak{N})$ is reduced then $\mathfrak{N}$ is reduced.*

Proof. Assume that $\mathfrak{N}$ is not reduced then there is a nonzero $x \in \mathrm{Ann}(\mathfrak{N})$ such that $B(x,x) = 1$. Since $[x, \mathfrak{N}] = [x, \mathfrak{N}]_+ = 0$ then $\mathfrak{g}(\mathfrak{N})$ and $\mathfrak{J}(\mathfrak{N})$ are not reduced. □

Corollary 4.33. *Let $\mathfrak{N}$ be a symmetric Novikov algebra. If $\mathfrak{g}(\mathfrak{N})$ is reduced then $\mathfrak{N}$ must be 2-step nilpotent.*

Proof. Since $\mathfrak{g}(\mathfrak{N})$ is reduced then $\mathrm{Ann}(\mathfrak{N}) \subset \mathfrak{N}\mathfrak{N}$. On the other hand, $\dim(C(\mathfrak{N})) = \dim([\mathfrak{N}, \mathfrak{N}]) = \frac{1}{2}\dim(\mathfrak{N})$ so $\dim(\mathrm{Ann}(\mathfrak{N})) = \dim(\mathfrak{N}\mathfrak{N})$. Therefore, $\mathrm{Ann}(\mathfrak{N}) = \mathfrak{N}\mathfrak{N}$ and $\mathfrak{N}$ is 2-step nilpotent. □

Example 4.34. By Example 4.2, every 2-step nilpotent algebra is Novikov then we will give here an example of symmetric non-commutative Novikov algebras of dimension 7 which is 3-step nilpotent. Let $\mathfrak{N} = \mathbb{C}x \oplus \mathfrak{N}_6$ be a 7-dimensional vector space, where $\mathfrak{N}_6$ is the symmetric Novikov algebra of dimension 6 in Example 4.26. Define the product on $\mathfrak{N}$ by

$$xe_4 = e_4x = e_1, e_4e_4 = x, e_4e_5 = e_3, e_5e_6 = e_1, e_6e_4 = e_2,$$

and the symmetric bilinear form B defined by

$$\begin{aligned} B(x,x) = B(e_1,e_4) = B(e_2,e_5) = B(e_3,e_6) = 1 \\ B(e_4,e_1) = B(e_5,e_2) = B(e_6,e_3) = 1, \\ 0 \ \text{ otherwise.} \end{aligned}$$

Note that in above Example, $\mathfrak{g}(\mathfrak{N})$ is not reduced since $x \in C(\mathfrak{N})$.

5. Appendix

Lemma 5.1. *Let (V,B) be a quadratic vector space, C be an invertible endomorphism of V such that*

(1) $B(C(x),y) = B(x,C(y)), \forall x,y \in V$.
(2) $3C - 2C^2 = \mathrm{Id}$

Then there is an orthogonal basis $\{e_1, \ldots, e_n\}$ of B such that C is diagonalizable with eigenvalues 1 and $\frac{1}{2}$.

Proof. Firstly, one has (2) equivalent to $C(C - \mathrm{Id}) = \frac{1}{2}(C - \mathrm{Id})$. Therefore, if x is a vector in V such that $C(x) - x \neq 0$ then $C(x) - x$ is an eigenvector with respect to eigenvalue $\frac{1}{2}$. We prove the result by induction on $\dim(V)$. If $\dim(V) = 1$, let $\{e\}$ be a orthogonal basis of V and assume $C(e) = \lambda e$ for some $\lambda \in \mathbb{C}$. Then by (2) one has $\lambda = 1$ or $\lambda = \frac{1}{2}$.

Assume that the result is true for quadratic vector spaces of dimension n, $n \geq 1$. Assume $\dim(V) = n+1$. If $C = \mathrm{Id}$ then the result follows. If $C \neq \mathrm{Id}$ then there exists $x \in V$ such that $C(x) - x \neq 0$. Let $e_1 := C(x) - x$ then $C(e_1) = \frac{1}{2}e_1$.

If $B(e_1,e_1) = 0$ then there is $e_2 \in V$ such that $B(e_2,e_2) = 0$, $B(e_1,e_2) = 1$ and $V = \mathrm{span}\{e_1,e_2\} \overset{\perp}{\oplus} V_1$, where $V_1 = \mathrm{span}\{e_1,e_2\}^{\perp}$. Since $\frac{1}{2} = B(C(e_1),e_2) = B(e_1,C(e_2))$ one has $C(e_2) = \frac{1}{2}e_2 + x + \beta e_1$, where $x \in V_1, \beta \in \mathbb{C}$. Let $f_1 := C(e_2) - e_2 = -\frac{1}{2}e_2 + x + \beta e_1$ then $C(f_1) = \frac{1}{2}f_1$ and $B(e_1,f_1) = -\frac{1}{2}$. If $B(f_1,f_1) \neq 0$ then let $e_1 := f_1$. If $B(f_1,f_1) = 0$ then let $e_1 := e_1 + f_1$. In the both cases, we have $B(e_1,e_1) \neq 0$ and $C(e_1) = \frac{1}{2}e_1$. Let $V = \mathbb{C}e_1 \overset{\perp}{\oplus} e_1^{\perp}$ then $e_1^{\perp}$ is non-degenerate, C maps $e_1^{\perp}$ into itself. Therefore the result follows the induction assumption. $\square$

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